The Clothoid

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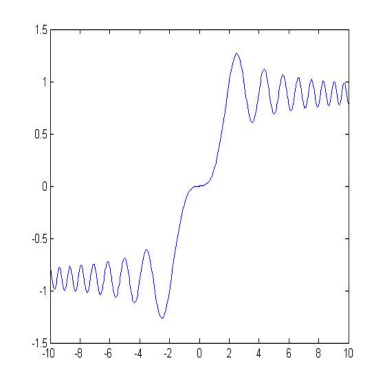
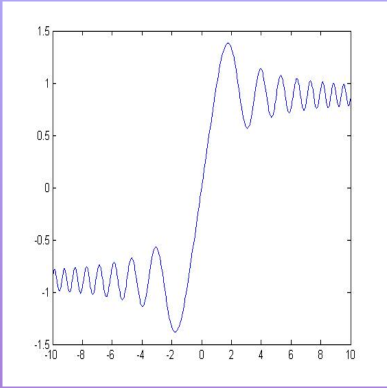
December 10, 2007

The Fresnel Functions

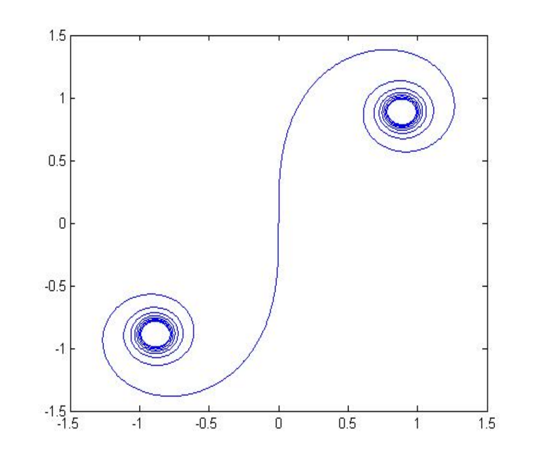


$$f(t) = \int_0^t sin\left(\frac{u^2}{2}\right)\ du$$



$$g(t) = \int_0^t cos\left(\frac{u^2}{2}\right)\ du$$

The Clothoid



$$\vec{\mathbf{r}}\,(t) = \left\langle \int_0^t sin\left(\frac{u^2}{2}\right)\ du, \int_0^t cos\left(\frac{u^2}{2}\right)\ du \right\rangle$$

Cesaro Equation

$$\kappa(s) = s$$

Dave's Kwik-E D.E.

Given $\kappa(s)$, show that a plane curve with curvature κ can be described by:

$$\vec{\boldsymbol{r}}(s) = \left\langle \int_0^s sin\theta(u)\ du, \int_0^s cos\theta(u)\ du \right\rangle$$

where $\theta(u) = \int_0^u \kappa(t)\ dt$.

Proof

$$
\begin{aligned}
\vec{\mathbf{r}}(s) &= \left\langle \int_0^s sin\ \theta(u)\ du, \int_0^s cos\ \theta(u)\ du \right\rangle \\
\hat{\mathbf{T}}(s) &= \frac{d\vec{\mathbf{r}}}{ds} \\
&= \langle sin\ \theta(s), cos\ \theta(s) \rangle \\
\left\| \frac{d\hat{\mathbf{T}}}{ds} \right\| &= \left\| \left\langle cos\ \theta(s) \frac{d\theta}{ds}, -sin\ \theta(s) \frac{d\theta}{ds} \right\rangle \right\| \\
&= \left| \frac{d\theta}{ds} \right| \| \langle cos\ \theta(s), -sin\ \theta(s) \rangle \| \\
&= \left| \frac{d\theta}{ds} \right| \\
&= \kappa(s)
\end{aligned}
$$

The Clothoid Revisited

$$\kappa(s) = s$$
$$\Rightarrow \theta'(s) = s$$

The Clothoid Revisited

$$\kappa(s) = s$$
$$\Rightarrow \theta'(s) = s$$
$$\int_0^t \theta'(s)\ ds = \int_0^t s\ ds$$
$$\theta(t) = \frac{t^2}{2}$$

The Clothoid Revisited

$$
\begin{aligned}
\kappa(s) &= s \\
\Rightarrow \theta'(s) &= s \\
\int_0^t \theta'(s)\ ds &= \int_0^t s\ ds \\
\theta(t) &= \frac{t^2}{2} \\
\vec{\mathbf{r}}(s) &= \left\langle \int_0^s sin\ \theta(u)\ du, \int_0^s cos\ \theta(u)\ du \right\rangle \\
\Rightarrow \vec{\mathbf{r}}(s) &= \left\langle \int_0^s sin\ \left(\frac{t^2}{2}\right)\ dt, \int_0^s cos\ \left(\frac{t^2}{2}\right)\ dt \right\rangle
\end{aligned}
$$

Circle with Radius 1

$$\kappa(s) = 1$$
$$\Rightarrow \theta'(s) = 1$$

Circle with Radius 1

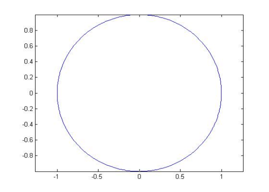
$$\begin{aligned} \kappa(s) &= 1 \\ \Rightarrow \theta'(s) &= 1 \\ \theta(t) &= t \end{aligned}$$

Circle with Radius 1

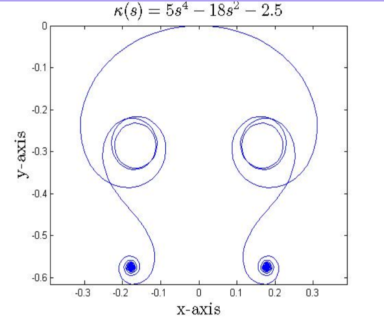
$$
\begin{aligned}
\kappa(s) &= 1 \\
\Rightarrow \theta'(s) &= 1 \\
\theta(t) &= t \\
\vec{\mathbf{r}}(s) &= \left\langle \int_0^s sin\ t\ dt, \int_0^s cos\ t\ dt \right\rangle \\
\vec{\mathbf{r}}(s) &= \langle -cos\ s, sin\ s \rangle
\end{aligned}
$$

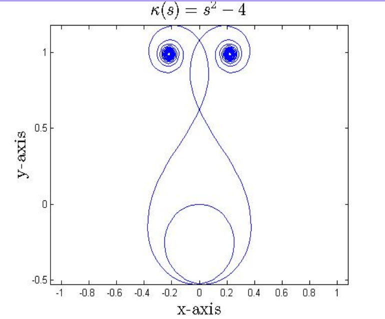
Circle with Radius 1

$$\kappa(s) = 1$$
$$\Rightarrow \theta'(s) = 1$$
$$\theta(t) = t$$
$$\vec{\mathbf{r}}(s) = \left\langle \int_0^s sin\ t\ dt, \int_0^s cos\ t\ dt \right\rangle$$
$$\vec{\mathbf{r}}(s) = \langle -cos\ s, sin\ s \rangle$$

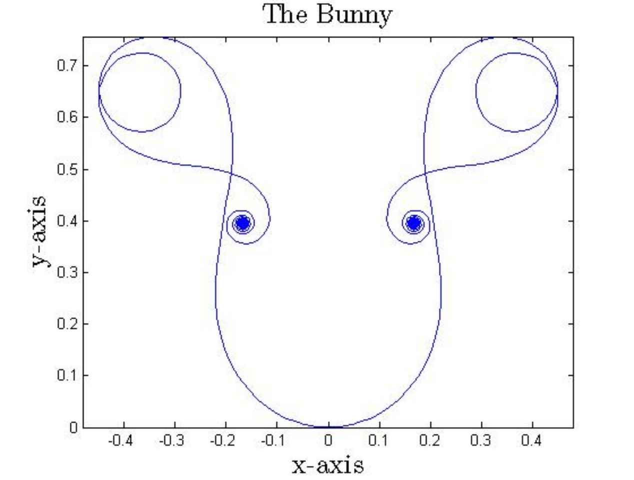


Alternate Spirals





Bunny



The Bunny
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
x-axis
y-axis