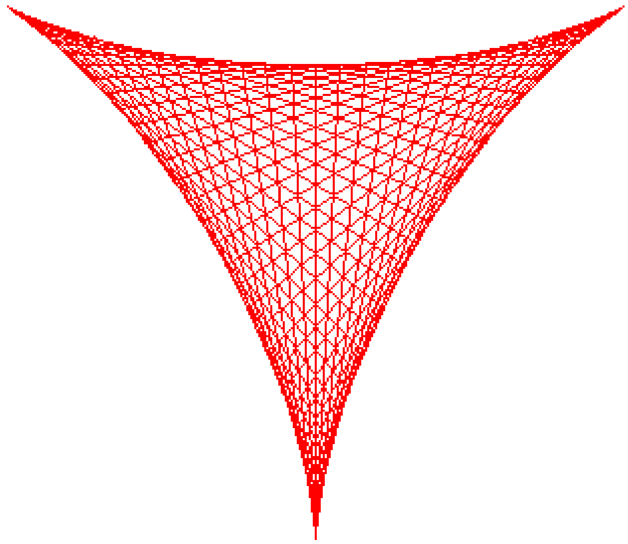

The Deltoid Curve

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Calculus 50C, Spring 1999

Description

The deltoid (also known as tricuspid, or Steiner's hypocycloid) is a member of the family of curves called Hypocycloids. A hypocycloid is a curve produced when a fixed point on a small circle is traced while the small circle rolls within a larger circle, and the deltoid curve is a three-cusped hypocycloid. The Deltoid curve is produced when the ratio of the radii of the circles is either 3/1 or 3/2 (the latter is called a double generation). Figure 2 shows the coordination of both methods. If this ratio is rational, the curve joins itself at the point of its origin. and the radius of the small circle is equal to the number of cusps that the curve has. If the ratio is irrational, the curve closes and fills the interior of the circle.

History of Deltoid

The deltoid was named for its resemblance to the Greek letter (delta). Although not having a single discoverer due to its relationship with the cycloid, Euler lays first claim to the deltoid in connection with an optical problem in 1745. Later, Steiner investigated the deltoid curve in 1856, adding his own contributions to it; it is now sometimes referred to as Steiner's Hypocycloid.

Equations and Properties

- Parametric Equations: (where a is the radius of the rolling circle)

$$x = a2\cos(\theta) + a\cos(2\theta)$$
$$y = a2\sin(\theta) - a\sin(2\theta)$$
$$0 \le \theta \le 2\pi$$

- Cartesian Equation:

$$(x^2 + y^2)^2 - 8x(x^2 - 3y^2) + 18(x^2 + y^2) - 27 = 0$$

- Length:

$$L = 16a$$

- Area:

$$A = 2\pi a^2$$

- Domain:

$$-\frac{3}{2}a \le x \le 3a$$

- Range:

$$-\frac{3\sqrt{3}}{2} \le y \le \frac{3\sqrt{3}}{2}$$

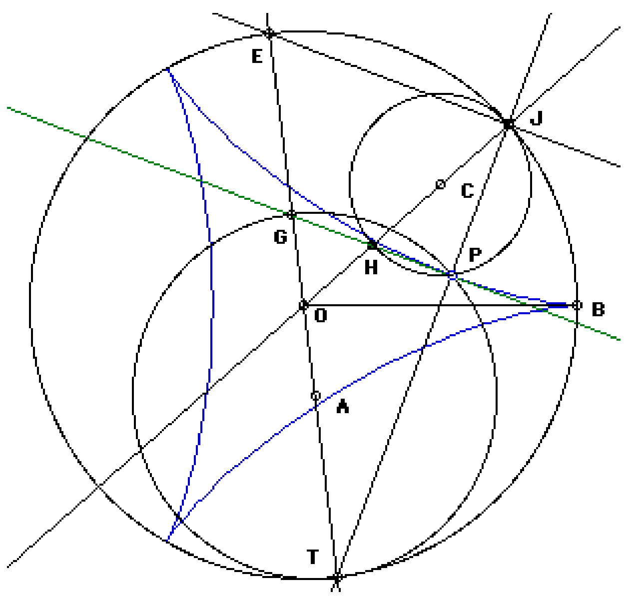

Figure 2: Deltoid Generated Illustration

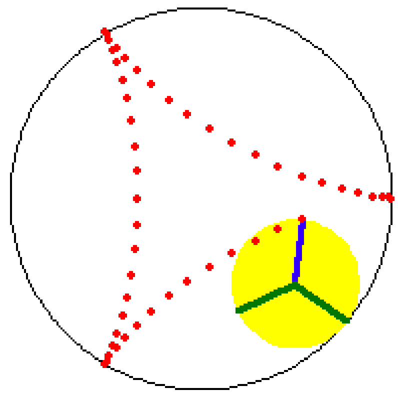

Figure 3: Hypotrochoid Deltoid

Parametrization of Deltoid Equation

A wheel of radius *CD* rolls counter-clockwise along the circumference of the larger circle. The radius of the fixed circle is either three or two-thirds as long as the wheel's radius. To find the parametric equations for the path (called the deltoid) traced by the point *F*, I need to find $F(x, y)$.

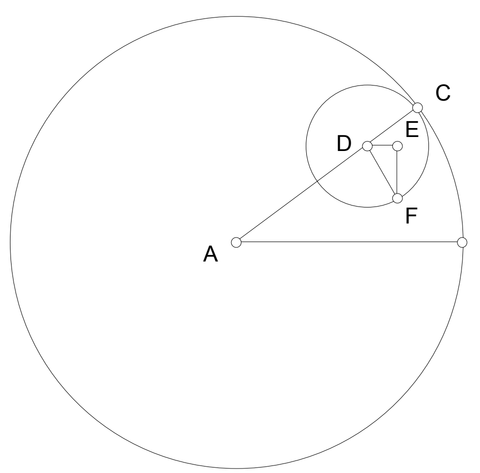

Figure 4.

Parametrization

Using Figure 4, we can make the following observations.

1. Make the radius *CD* of the wheel be $\frac{1}{3}$ of the radius of the larger circle.
2. The length of *AD* will be 3 times as long as *CD* (written 3*CD*).
3. The line *AD* has length 2*CD*.
4. The arc length *BC* will be $3CDt$, and the arc length of *FC* is $3CDt$.
5. Therefore, the coordinates of point *D* is

$$(2a\cos(t), 2a\sin(t)).$$

6. The angle between CD and DE is t.
7. Arc length of BC divided by the length CD produces the angle between AD and DF;

$$\frac{3at}{a} = 3t$$

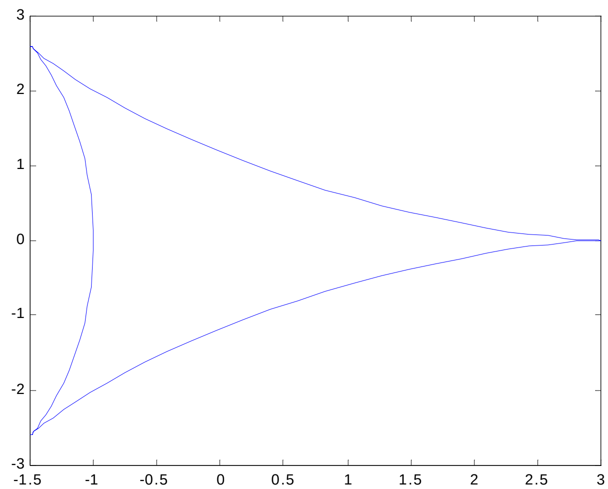

Figure 5: Matlab Generated Deltoid

8. The angle $3t$ subtracted from the angle t gives the angle between DE and DF.
9. Angle FDE: $3t - t = 2t$.
10. Length of DE: $CD\cos(2t)$.
11. Length of EF: $CD\sin(2t)$.
12. Combining I get

$$x = 2CD\cos(t) - CD\cos(2t)$$

$$y = 2CD\sin(t) - CD\sin(2t).$$

13. $CD = a$, and $t = \theta$
14. Replacing variables I get,

$$x = 2a\cos(\theta) - a\cos(2\theta)$$

$$y = 2a\sin(\theta) - a\sin(2\theta).$$

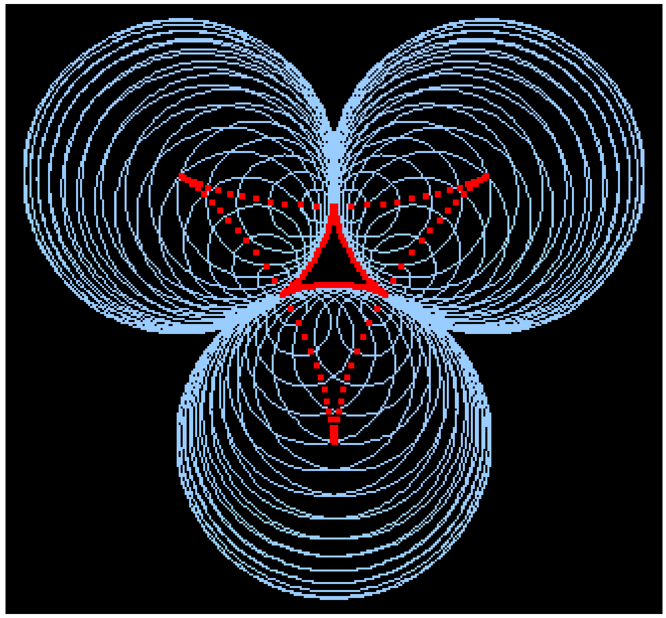

Figure 6: Deltoid constructed by Oscillating Circles

Now we get to the fun part of this project where we get to see a deltoid constructed from a geometer's sketchpad drawing. You can access the file `Delta4.gsp` at

`http:\\online.redwoods.cc.ca.us\instruct\darnold\CalcProj\Sp99\Ed\Delta4.gsp`

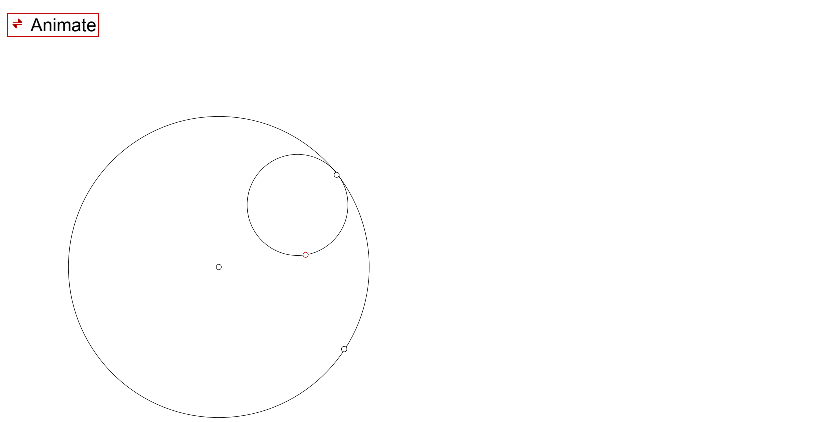

Figure 7: Goemeter's Sketchpad Animation

References

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• Yates, Robert. Curves And Their Properties. pp71-74, 1952