

# The Nephroid

Russ Hooper  
College of the Redwoods  
Spring 1999

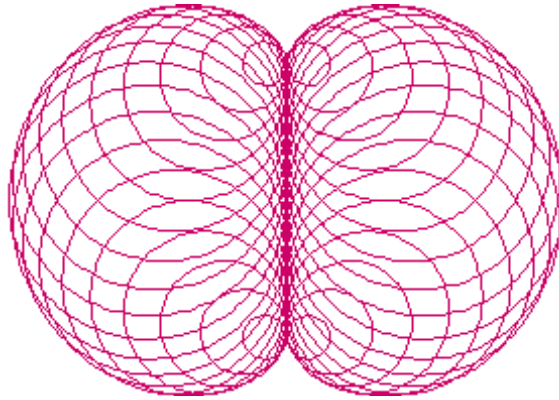


Figure 1. Envelope of Circles

## History

First studied in 1697 by Huygens and Tshirnhauser, the Nephroid was shown to be the caustic of a circle when the light source is at infinity. The theory of caustics is a method of finding a new curve based on a given curve and a specified point. Later, in 1692 Bernoulli showed that the nephroid is the caustic of a cardioid for a luminous cusp. A caustic is the trace of rays from a given point reflecting off a curve. However, the name *Nephroid* was not actually used until the English mathematician R.A. Proctor used the term in "The Geometry of Cycloids." Nephroid, meaning "kidney-shaped" was used by Proctor to describe a two cused epicycloid.

## Description

The nephroid is a special case of the epicycloid. Nephroid can be defined as the trace of a point fixed on a circle of radius  $1/2r$  that rolls about the circumference of a fixed circle with  $r$ . The path traced by this point forms the nephroid.

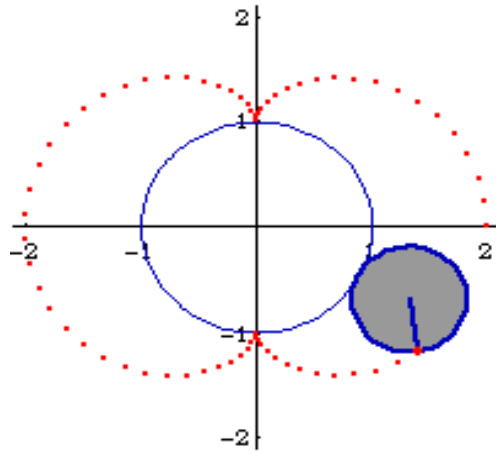


Figure 2. The Nephroid

### Constructing the Nephroid

Let  $A$  be the center of the fixed circle with radius  $r$ . Let  $B$  be the center of the rolling circle with radius  $1/2r$ . Let  $P$  be the tracing point.  $D$  will be the contact point between the circles.  $PDH$  are colinear and  $HACJ$  are collinear, where  $C$  is  $1/2r$  from  $A$ . Let  $J$  be the reflection of  $H$  through  $C$ . Then  $C$  forms the center of the larger rolling circle with radius  $CH = 3/2r$ .  $JPI$  is collinear and is the tangent at point  $P$ .  $PD$  is the normal at  $P$ . Let  $Q$ , which is not shown, be the point formed by rotating  $P$  around  $B$  by 180 degrees. The intersection of  $AQ$  and  $PD$  creates the center of the osculating circle at  $P$ . The line from cusp to cusp is the diameter of the nephroid.

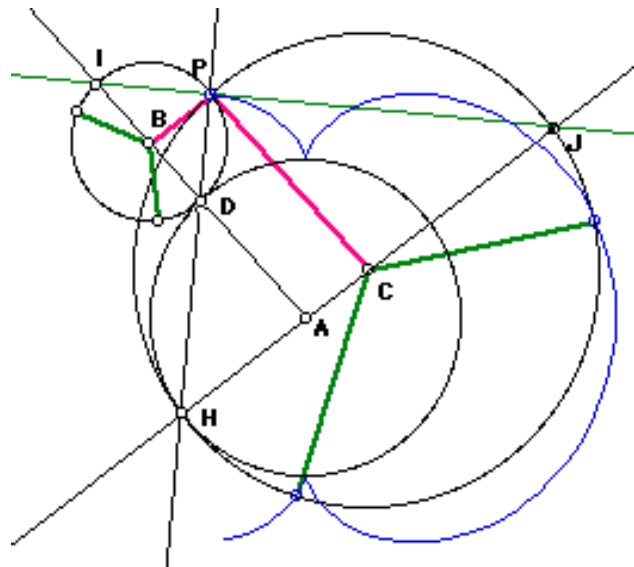


Figure 3. Nephroid Construction

### Formulas

- Distance from cusp to cusp:  $4a$

- Length of curve:  $24a$
- Area of region enclosed by curve:  $12\pi a^2$
- Surface area of revolution:  $\frac{128}{5}\pi a^2$

## Developing System of Parametric Equations

The Parametric Equations that define a nephroid will be shown to be

$$x = a(3 \cos(t) - \cos(3t))$$

$$y = a(3 \sin(t) - \sin(3t))$$

**Development of equations:**

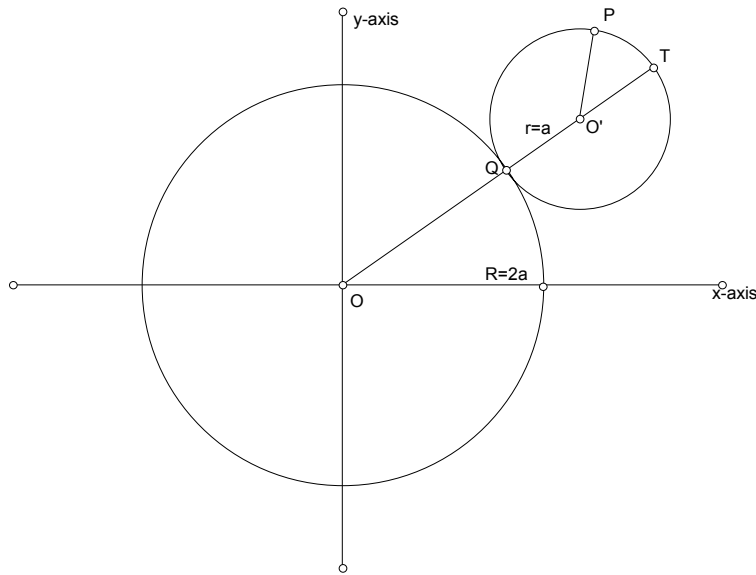


Figure 4.

The center of the large circle is O, the center of the small circle is O'. P is the tracing point, Q is the point of contact between circles. Let's make angle XOQ equal to  $t$ . The coordinates of O' will have to be  $(3a \cos(t), 3a \sin(t))$ . Then angle PO'T =  $2t$ , and the inclination of O'P to OX is  $3t$ . The coordinates of P  $(x, y)$  are then given by the following equations,

$$x = 3a \cos(t) + a \cos(3t)$$

$$y = 3a \sin(t) + a \sin(3t)$$

And for the point opposite to P on the rolling circle is therefore given by the parametric equations,

$$x = 3a \cos(t) - a \cos(3t)$$

$$y = 3a \sin(t) - a \sin(3t)$$

The parametric equations can also be derived given the equation of an epicycloid,

$$x = (a + b) \cos(t) - b \cos((a/b + 1)t)$$

$$y = (a + b) \sin(t) - b \sin((a/b + 1)t)$$

Since  $b = 1/2a$  in a nephroid, then

$$x = (a + (1/2)a) \cos(t) - (1/2)a \cos((1/2)a + 1)t$$

$$y = (a + (1/2)a) \sin(t) - (1/2)a \sin((a/((1/2)a) + 1)t)$$

$$x = (3/2)a \cos(t) - (1/2)a \cos((2 + 1)t)$$

$$y = (3/2)a \sin(t) - (1/2)a \sin((2 + 1)t)$$

$$x = (1/2)a(3 \cos(t) - \cos(3t))$$

$$y = (1/2)a(3 \sin(t) - \sin(3t))$$

A is then substituted for  $1/2a$ , and thus the parametric equations,

$$x = A(3 \cos(t) - \cos(3t)),$$

$$y = A(3 \sin(t) - \sin(3t)).$$

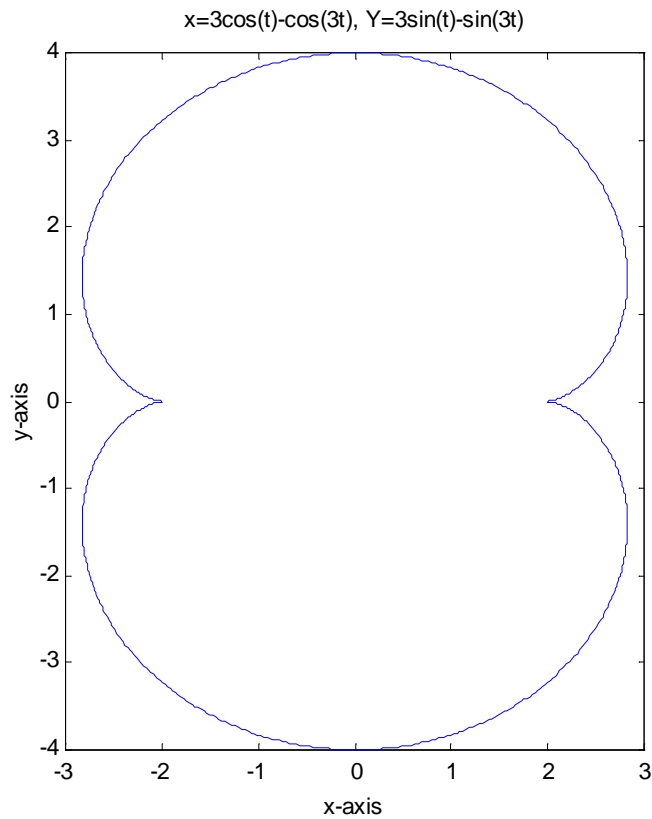


Figure 5. Matlab image of Nephroid by parametric equations

## Special Properties

The nephroid also has some other interesting properties. It is the caustic of a

circle with parallel rays and the caustic of a cardioid with the lightsource at the cusp. The nephroid has half the area of the cardioid (see Figure 6).

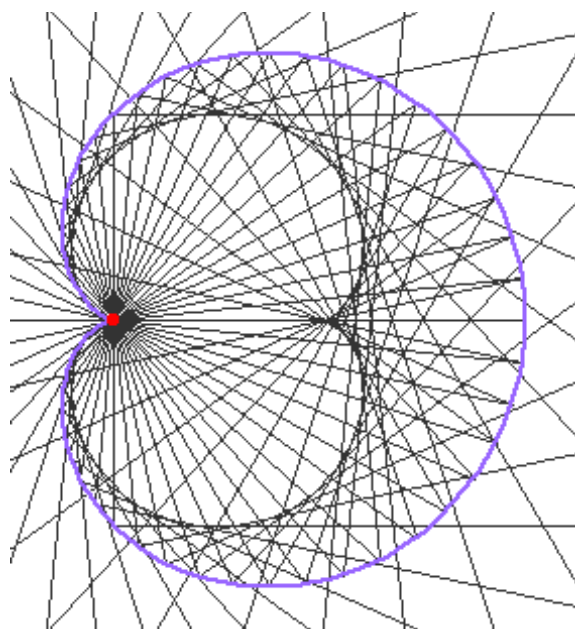


Figure 6. Nephroid by Cardioid

Probably the coolest property is that when parallel light rays bounce off the sides of an empty bottomed coffee mug, a nephroid is formed!

### Sources

1. *Nephroid*, [http://www.best.com/~xah/SpecialPl...ves\\_dir/Nephroid\\_dir/nephroid.html](http://www.best.com/~xah/SpecialPl...ves_dir/Nephroid_dir/nephroid.html)
2. *Nephroid*, <http://www-groups.dcs.st-and.ac.uk/~history/Curves/Nephroid.html>
3. *A Book of Curves*, E.H. Lockwood, Cambridge University Press