Lemniscate of Bernoulli

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Multivariable Calculus

Abstract. A description and definition of the Lemniscate of Bernoulli that leads to the development of it's parametric equation, along with a brief history of the curve.

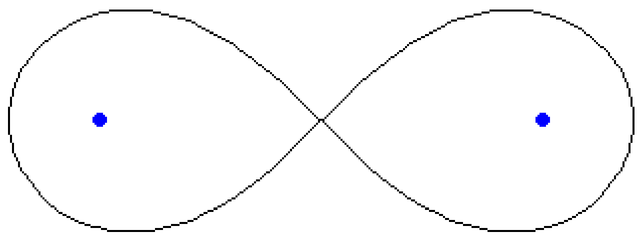

Figure 1. (Xah Lee)

History

In 1694 James Bernoulli, a member of the famous Bernoulli family of mathematicians published his findings on a curve which he called a lemniscus. A lemniscus is Latin for the word ribbon. This curve is a special case of a Cassinian Oval and it's arclength became very important for later work on elliptical functions. Another interesting fact about the lemniscate is that it is symmetric about the *x*-axis, the *y*-axis, and the origin.

Definition *The Lemniscate is defined as the locus of a point, the product of whose distances from two fixed points* $(-a,0)$*and*$(a,0)$, *the foci*, *is* $2a$ *units apart and is equal to* a^2

Cartesian Formula

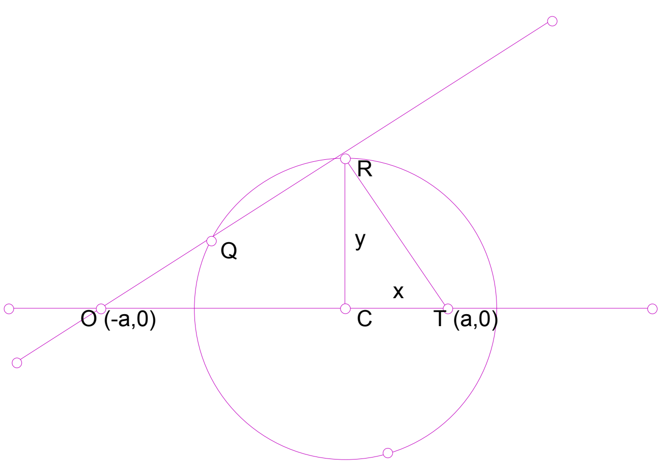

Figure 2. Geometer's Sketchpad

$$
\begin{aligned}
(OR)^2 &= y^2 + (x+a)^2 \\
(RT)^2 &= y^2 + (x-a)^2 \\
(OR)^2(RT)^2 &= a^2 \\
(y^2 + (x+a)^2)(y^2 + (x-a)^2) &= a^2 \\
(y^2 + x^2 + 2xa + a^2)(y^2 + x^2 - 2xa + a^2) &= a^2 \\
(x^2 + y^2)^2 &= a^2(x^2 - y^2)
\end{aligned}
$$

Parametric Formula

$$
\begin{aligned}
x &= r\sin\theta \\
y &= r\cos\theta \\
x^2 + y^2 &= r^2
\end{aligned}
$$

$$
\begin{aligned}
(x^2 + y^2)^2 &= a^2(x^2 - y^2) \\
r^4 &= r^2\cos^2\theta - r^2\sin^2\theta \\
r^2 &= \cos^2\theta - \sin^2\theta \\
r^2 &= \cos 2\theta \\
r &= \pm\sqrt[2]{\cos 2\theta} \\
x &= \cos\theta \pm \sqrt[2]{\cos 2\theta}\,;\ y = \sin\theta \pm \sqrt[2]{\cos 2\theta}
\end{aligned}
$$

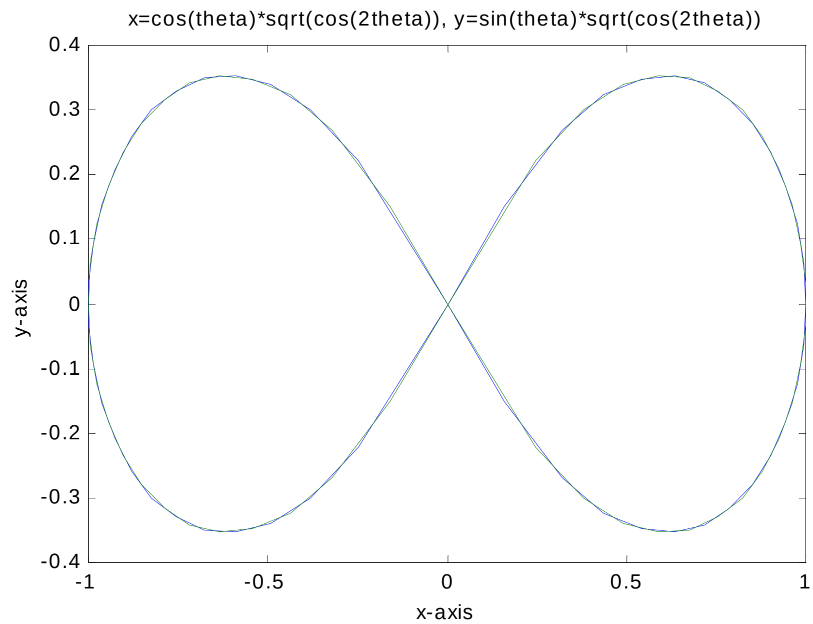

Figure 3. Matlab Generated

Other Properties

Cissoid

The Lemniscate can be described as a cissoid of two circles with a vector that origin is O. The locus of this vector is one loop of Bernoulli's Lemniscate.

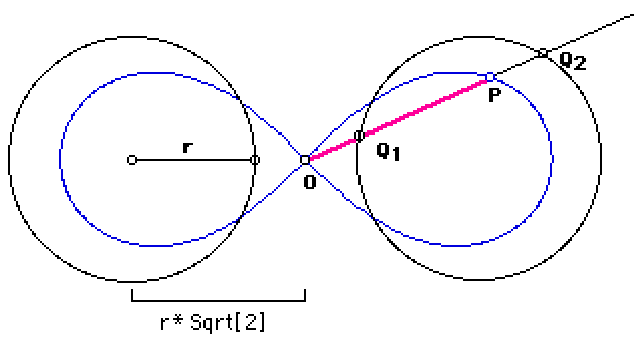

Figure 4. (Xah Lee)

Hyperbola

The Lemniscate can also be described as "the envelop of circles with centers on a rectangular hyperbola and each circle passing the hyperbola's center" (Xah Lee).

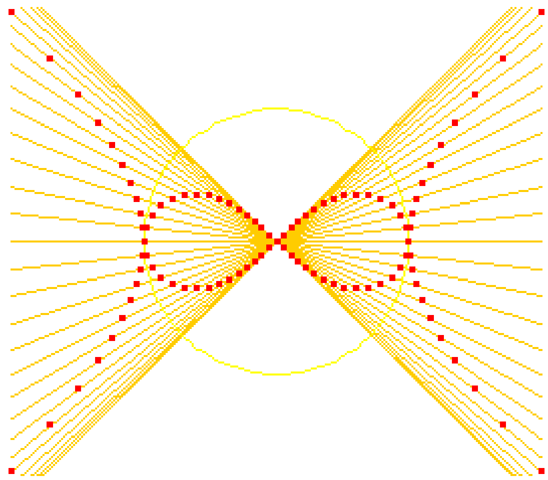

Figure 5. (Xah Lee)

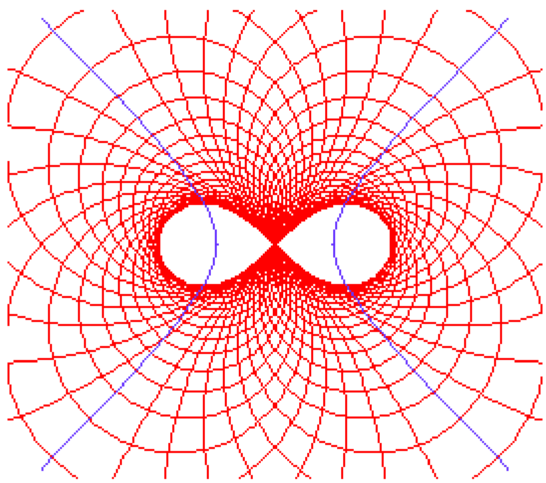

Figure 6. (Xah Lee)

Slicing a Torus

The last way that I will describe the Lemnicate, although there are many others, is slicing a torus. Such that the "Lemniscate of Benoulli is the intersection of a plane tangent to the inner ring of a torus whose inner radius equals to its radius of generating circle" (Xah Lee).

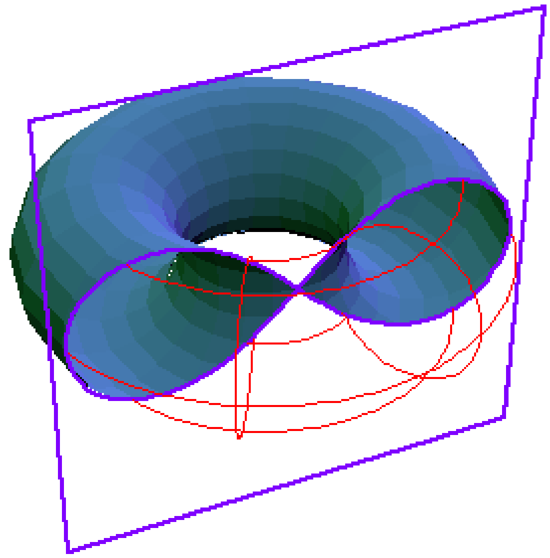

Figure 7. (Xah Lee)

References

- Lee, Xah. www.best.com. Website of Special Plane Curves
- Lockwood, L.H. A Book of Curves. Cambridge University Press. 1963. (112-117).
- Yates, Robert C. Curves and Their Properties. National Council of Teachers of Mathematics. 1974. (143-147).