The Deltoid

Ken Huffman

May 18, 2005

1 History

The Deltoid is a special case of the hypocycloid. A Deltoid is a three cusped hypocycloid, also called a tricuspoid. There is very little information on the history of the Deltoid. But Euler was the first to work with the Deltoid in 1745. He was using the Deltoid in connection with an optical problem, but not much more is known about this. Steiner studied the Deltoid later in 1856, and it is commonly referred to as Steiner's hypocycloid.

A hypocycloid is a large fixed circle, with a smaller rolling circle inside. For the Deltoid the smaller circle must have a radius one third or two thirds the larger circle. If the inner circle has a radius one third the larger circle, it traces out the Deltoid in one trip around the larger circle. If the inner circle has a radius two thirds the larger circle. The point traces out the Deltoid twice in one trip around the larger circle. This is commonly called double generation.

2 Parameterizing the Deltoid

2.1 Ratio 3:1

Let's begin by defining a circle of radius a. Therefore the radius of the small circle is $a/3$. If we let θ be the angle defined by $\angle BCD$, then the arc length of the segment between B and D is given by the formula

$$S_{BD} = a\theta. \tag{1}$$

Similarly, if we let ϕ be the angle on the small circle defined by $\angle DOP$. The arc length between D and P is given by the formula

$$S_{DP} = \frac{a}{3}\phi. \tag{2}$$

If we assume there is no slippage when the small circle roles inside of the large circle. Then the arc length from D to P is equal to the arc length from B to D. Using this

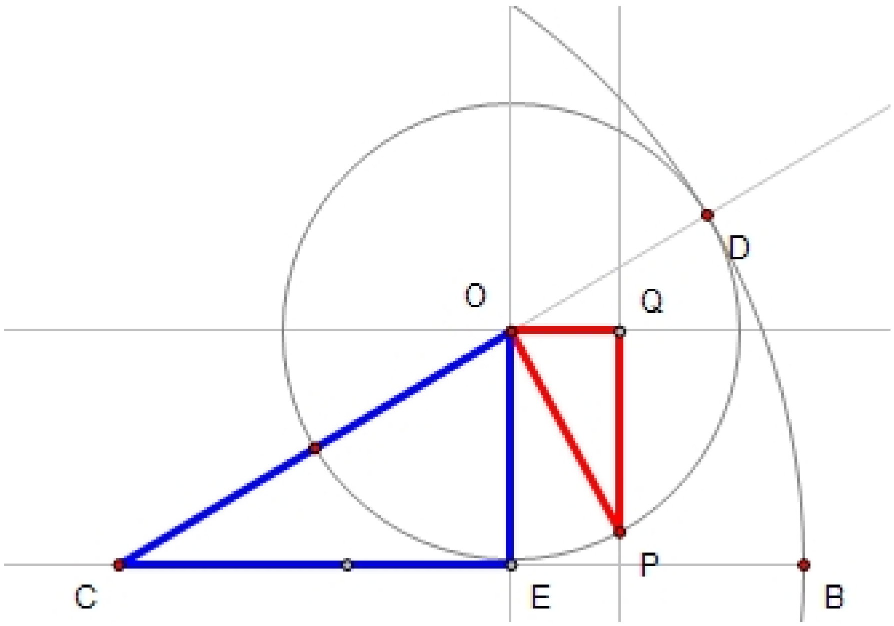

Figure 1: Parameterizing the Deltoid with 3:1 ratio.

information we can equate the angle ϕ with the angle θ

$$\frac{a}{3}\phi = a\theta \tag{3}$$

$$\phi = 3\theta. \tag{4}$$

In order to parameterize the motion of the circles we need to draw some triangles. Please refer to Figure 1 . If we draw a vertical line down through the point O we obtain the first triangle we need. This triangle is bounded by the points C, O, and E. If we draw a line parallel to the segment $\overline{CB}$ though the point O. Then draw a vertical line though P. This gives us the second triangle we need. It is bounded by the points O, P, and Q. The angle between the segment $\overline{OQ}$ and the segment $\overline{OD}$ is θ, by similar triangles. We know the angle ϕ is equivalent to 3θ, therefore the angle $\angle QOP$ is 2θ

Using this information we can determine the x and y coordinates in terms of θ. The x coordinate can be obtained by adding the segment $\overline{CE}$ and the segment $\overline{OQ}$. Using a little trigonometry, we see that

$$x = \frac{2a}{3}\cos\theta + \frac{a}{3}\cos 2\theta. \tag{5}$$

In the same manner we can obtain the y coordinate by subtracting the segment $\overline{PQ}$ from the segment $\overline{OE}$. Giving us

$$y = \frac{2a}{3}\sin\theta - \frac{a}{3}\sin 2\theta. \tag{6}$$

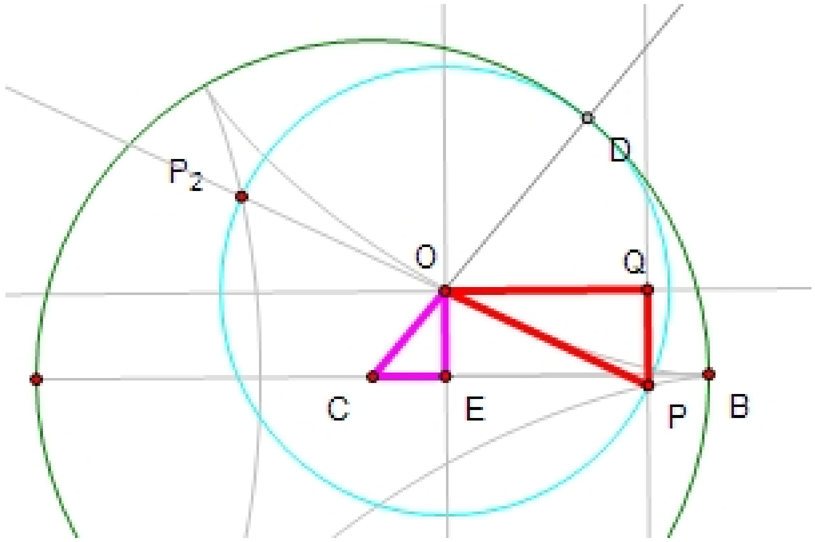

Figure 2: Parameterizing the Deltoid with 2:3 ratio.

Where $0 \leq \theta \leq 2\pi$.

2.2 Ratio 3:2

These equations will be different if the circles have the ratio of 3:2. Let's begin as before, by defining a circle of radius a. Therefore the radius of the small circle is $2a/3$. As before if we let θ be the angle defined by $\angle BCD$, and we let ϕ be the angle on the small circle defined by $\angle DOP$. Then the relation between θ and ϕ is given by

$$\frac{2a}{3}\phi = a\theta \tag{7}$$

$$\phi = \frac{3}{2}\theta. \tag{8}$$

In order to parameterize the motion of the circles we need to draw some triangles. Please refer to Figure 2 . If we draw a vertical line down through the point O we obtain the first triangle we need. This triangle is bounded by the points C, O, and E. If we draw a line parallel to the segment $\overline{CB}$ though the point O. Then draw a vertical line though P. This gives us the second triangle we need. It is bounded by the points O, P, and Q. The angle between the segment $\overline{OQ}$ and the segment $\overline{OD}$ is θ, by similar triangles. We know the angle ϕ is equivalent to $(3\theta/2)$ therefore the angle $\angle QOP$ is $(\theta/2)$

Using this information we can determine the x and y coordinates in terms of θ. The x coordinate can be obtained by adding the segment $\overline{CE}$ and the segment $\overline{OQ}$. Using a little trigonometry, we see that

$$x = \frac{a}{3}\cos\theta + \frac{2a}{3}\cos\frac{\theta}{2}. \tag{9}$$

In the same manner we can obtain the y coordinate by subtracting the segment $\overline{PQ}$ from the segment $\overline{OE}$. Giving us

$$y = \frac{a}{3}\sin\theta - \frac{2a}{3}\sin\frac{\theta}{2}. \tag{10}$$

Where $0 \leq \theta \leq 2\pi$.

3 Useful Formulas

There is a large number of useful formulas that are associated with the Deltoid. The arc length of the Deltoid is given by the formula

$$s(t) = \frac{16}{9}\sin^2\left(\frac{3t}{4}\right). \tag{11}$$

The curvature of the Deltoid is given by the formula

$$\kappa(t) = -\frac{3}{8}\csc\left(\frac{3t}{2}\right). \tag{12}$$

The total arc length can be computed using the arc length formula of the general hypocycloid. Which is given by

$$s_{tot} = \frac{8a(n-1)}{n}. \tag{13}$$

Where n is the ratio of the outer circle to the inner circle. For example, the Deltoid with the 3:1 ratio has a total arc length of

$$s_{tot} = \frac{8a(3-1)}{3} \tag{14}$$

$$s_{tot} = \frac{16a}{3}. \tag{15}$$

The area inside the Deltoid is given by the formula

$$A_{enc.} = \frac{(n-1)(n-2)}{n^2}\pi a^2. \tag{16}$$

Again, using the Deltoid with the 3:1 ratio. The total area enclosed by the Deltoid is

$$A_{enc.} = \frac{(3-1)(3-2)}{3^2}\pi a^2 \tag{17}$$

$$A_{enc.} = \frac{2}{9}\pi a^2. \tag{18}$$

4 Geometers Sketchpad

Both the 3:1 and 3:2 Deltoids are available for experimentation at
http://online.redwoods.edu/instruct/danold/Calcproj/sp05/khuffman/3_1_Deltoid.gsp
and
http://online.redwoods.edu/instruct/danold/Calcproj/sp05/khuffman/3_2_Deltoid.gsp.

These Geometers Sketchpad files are a great interactive learning tool. Click on the animate button and watch the Deltoid being traced out by the point P. In order to sketch the full Deltoid with the 3:2 ratio circles, it is necessary to define another point on the smaller radius circle. This point is opposite the point P and is labelled P_2. Click on the animate button and the Deltoid will be traced out by the points P and P_2.

References

[1] E.H. Lockwood. A Book of Curves. Cambridge University Press, 1961.

[2] None. *"Deltoid."* (2005):n.pag. Online. Internet. 30 Apr. 2005. Available: http://mathworld.wolfram.com/Deltoid.html.

[3] Darling, David. *"The Encyclopedia of Astrobiology, Astronomy, and Spaceflight."* (2005):n.pag. Online. Internet. 30 Apr. 2005. Available: http://www.daviddarling.info/encyclopedia/D/deltoid.html.

[4] Lee, Xah. *"Deltoid."* (2005):n.pag. Online. Internet. 30 Apr. 2005. Available: http://www.xahlee.org/SpecialPlaneCurves_dir/Deltoid_dir/deltoid.html.