Humboldt State University
Mathematics Department
Math 115—Precalculus

Exam #3—Trigonometry

David Arnold

Last Revision Date: December 11, 2000 Version 1.00

Essay Questions

Instructions: *Place the solution to each of the following problems on a separate sheet of paper (you may use the front and back sides). It is required that you do 7 of the following 9 questions, no more, no less. When you complete the exam, arrange your solutions in order, then staple the exam on top as a cover page. Place your name on the exam. You will be docked severely for not following instructions. For example, if you put two or more problems on one sheet of paper, there will be a deduction. If you hand in 8 problems instead of 7, then I will correct 7 of my choice. Rest assured that one of my choices will be the one you least wanted graded. If your solutions are not arranged in order, there will be a deduction. If your name is not on the cover sheet, there will be a deduction, etc., etc. Follow directions or lose points! With that said, good luck!*

EXERCISE 1. A surveying crew is given the job of measuring the height of a mountain. From a point on level ground, they measure the angle of elevation to the top of the mountain at $21°$. They move 507 meters closer and find the angle of elevation to the top of the mountain is now $35°$. How high is the mountain?

EXERCISE 2. Tarzan is swinging back and forth on his grapevine. As he swings, he goes back and forth across the river bank, going alternatively over land and water. Jane decides to model mathematically his motion and starts her stop watch. Let t be the number of seconds the stopwatch reads and let y be the number of meters Tarzan is from the river bank. Assume that y varies sinusoidally with t, and that y is positive when Tarzan is over water and negative when he is over land. Jane finds that when $t = 2$, Tarzan is at one end of his swing, where $y = -23$. She finds that when $t = 5$, he reaches the other end of his swing and $y = 17$.

(a) Sketch a graph of this function.

(b) Write an equation expressing Tarzan's distance from the river bank in terms of t.

(c) Where was Tarzan when Jane started the stopwatch?

EXERCISE 3. Prove any two of the following three identities.

(a) $\dfrac{\tan\theta + \cot\theta}{\sec\theta\csc\theta} = 1$

(b) $\dfrac{\sin\theta}{1+\cos\theta} = \dfrac{1-\cos\theta}{\sin\theta}$

(c) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

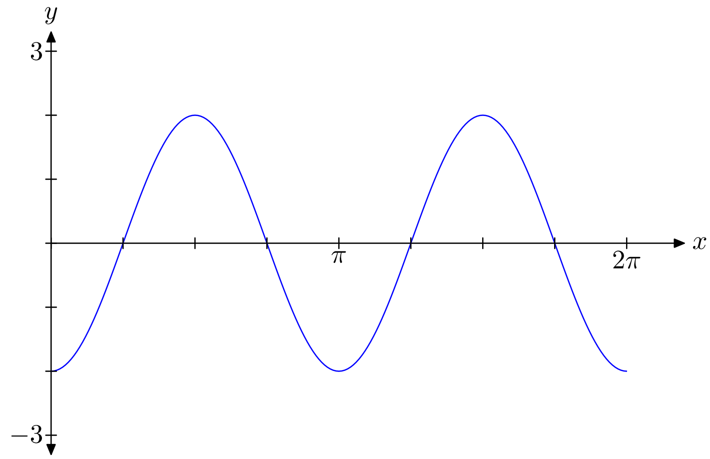
EXERCISE 4. Solve any two of the following three equations on the interval $[0, 2\pi)$. I am not interested in solutions found by a calculator. Rather, I want to see all appropriate work: factoring, identities, etc. Also, please give exact answers only. Decimal approximations will receive no credit.

(a) $2\sin^2 x - 3\sin x + 1 = 0$

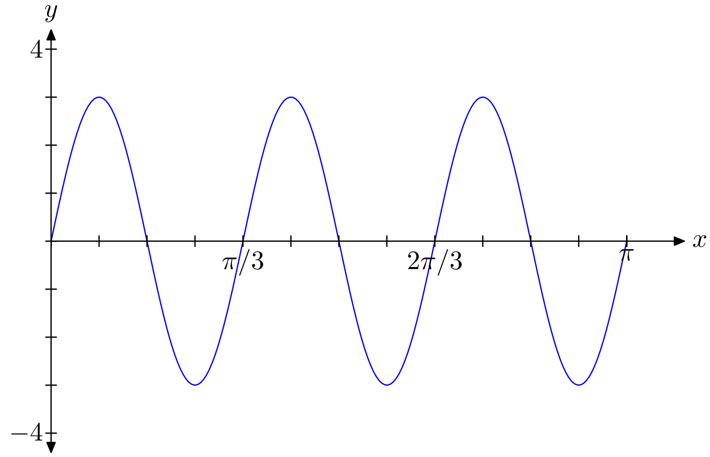
(b) $\tan^2 2x = 3$

(c) $\sin 2x = \cos x$

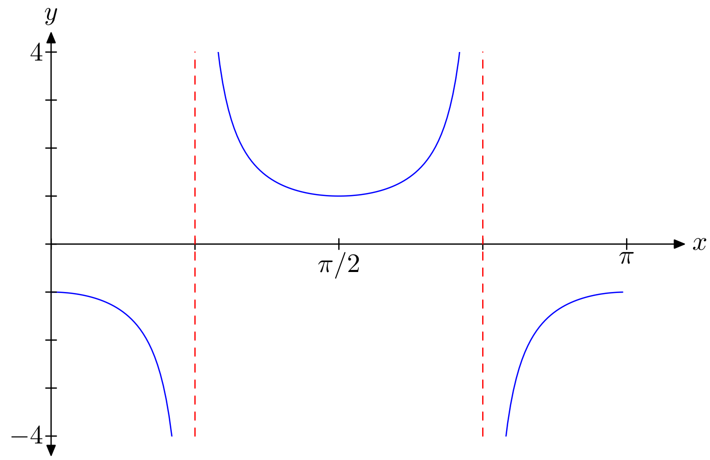
Exercise 5. Answer any two of the following three questions. In each case, state the amplitude, period, and phase shift (if any), then give the equation of the given graph.



(a)



(b)



(c)

Exercise 6. Answer each of the following questions.

(a) Sketch the graph of $y = \sin^{-1} x$. State the domain and range using interval notation. No decimal approximations will be accepted.

(b) Evaluate $\cos(\tan^{-1}(-2))$. Show all appropriate work and place answer in simple radical form. No credit will be given for any decimal approximations or answers without supportive work.

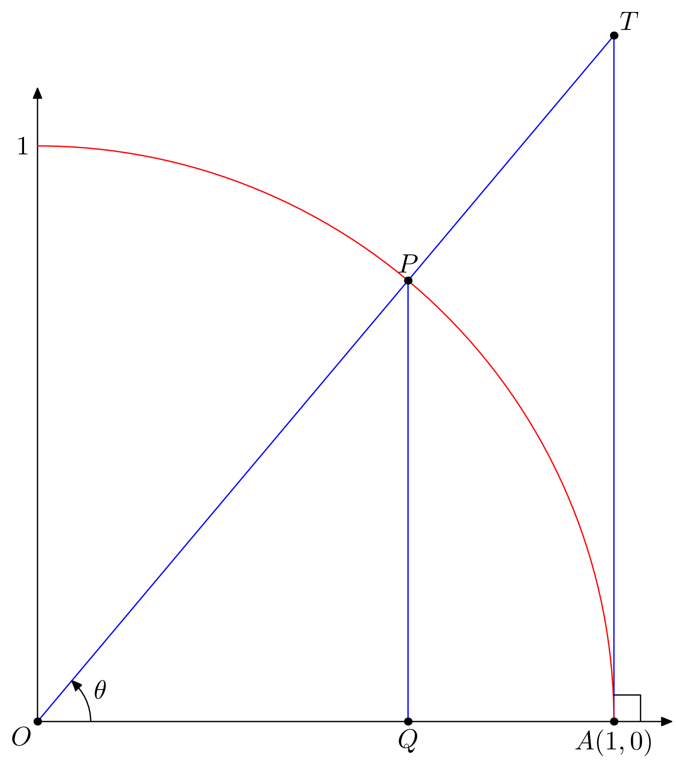
Exercise 7. Answer any two of the following three questions. No credit will be given for any decimal approximations or solutions without supportive work.

(a) If $\sin x = 4/5$, $\pi/2 < x < \pi$, and $\sin y = -5/12$, $\pi < y < 3\pi/2$, find the exact value of $\sin(x + y)$.

(b) If $\cos x = -2/3$, $\pi/2 < x < \pi$, find the exact value of $\cos 2x$.

(c) Find an exact value of $\sin \pi/8$ in simple radical form.

Exercise 8. Consider the following figure, which contains one-quarter of the unit circle in the first quadrant.



(a) Find length OQ and QP in terms of θ.

(b) What is the length of arc $\widehat{AP}$?

(c) Find the length AT in terms of θ.

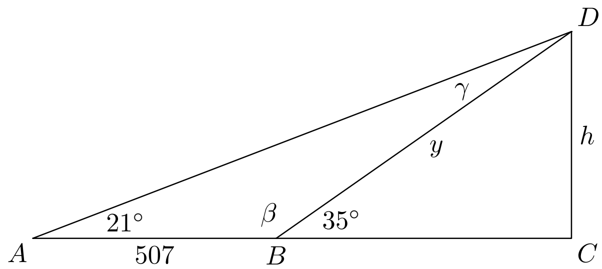
Exercise 9. A ferris wheel 60 ft in diameter is turning in a counterclockwise direction. The wheel makes 1 revolution every 20 seconds.

(a) Find the angular velocity in radians per second.

(b) Find the speed of a rider on the circumference of the wheel in ft/s.

Solutions to Exercises

Exercise 1. A diagram is helpful.



Angle β and angle $\angle DBC$ are supplementary (their sum is 180°). Thus, $\beta = 145°$. Because the angles of a triangle sum to 180°,

$$\gamma = 180° - 21° - 145°$$
$$\gamma = 4°.$$

Now, we can use the law of cosines to find y.

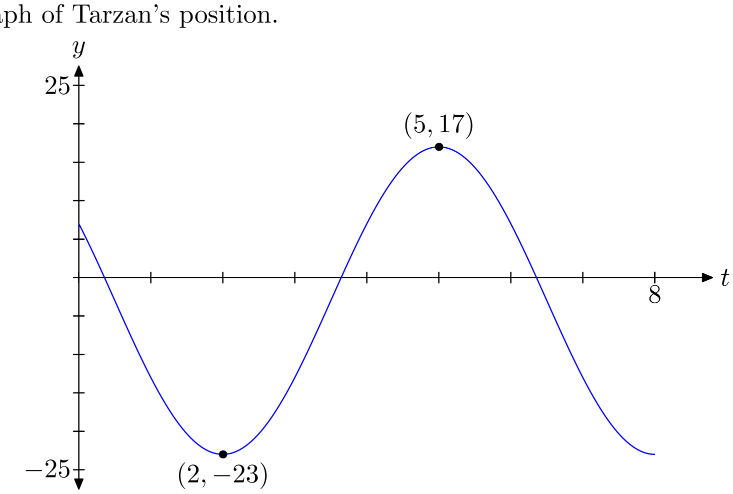
$$\frac{\sin 21°}{y} = \frac{\sin 14°}{507}$$
$$y \sin 14° = 507 \sin 21°$$
$$y = \frac{507 \sin 21°}{\sin 14°}$$
$$y = 751$$

We can now solve the right triangle $\triangle BCD$ for h, the height of the mountain.

$$\sin 35° = \frac{h}{751}$$
$$h = 751 \sin 35°$$
$$h = 431 \text{ meters}$$

Exercise 1

Exercise 2(a) A graph of Tarzan's position.



□

Exercise 2(b) The amplitude is 20. The period is $T = 6\,\text{s}$. This can be used to find B.

$$\begin{aligned} T &= \frac{2\pi}{B} \\ 6 &= \frac{2\pi}{B} \\ B &= 2\pi \\ B &= \frac{2\pi}{6} \\ B &= \frac{\pi}{B} \end{aligned}$$

In the figure, I am seeing an inverted cosine that is translated 2 seconds to the right and 3 meters down. Thus,

$$y = -20\cos\frac{\pi}{3}(t-2) - 3.$$

□

Exercise 2(c) When Jane started her watch, the time was $t = 0$. Substitute $t = 0$ into the equation for Tarzan's position.

$$\begin{aligned} y &= -20\cos\frac{\pi}{3}(0-2) - 3 \\ y &= -20\cos\left(-\frac{2\pi}{3}\right) - 3 \\ y &\approx 7\,\text{m} \end{aligned}$$

□

Exercise 3(a) Proof;

$$\begin{aligned} \frac{\tan\theta + \cot\theta}{\sec\theta\csc\theta} &= \frac{\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta}}{\dfrac{1}{\cos\theta}\cdot\dfrac{1}{\sin\theta}} \\ &= \frac{\left(\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta}\right)(\cos\theta\sin\theta)}{\left(\dfrac{1}{\cos\theta\sin\theta}\right)(\cos\theta\sin\theta)} \\ &= \frac{\sin^2\theta + \cos^2\theta}{1} \\ &= \frac{1}{1} \\ &= 1 \end{aligned}$$

□

Exercise 3(b) Proof: Multiply numerator and denominator by $1 - \cos\theta$.

$$\begin{aligned} \frac{\sin\theta}{1 = |cos\theta} &= \frac{\sin\theta(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)} \\ &= \frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta} \end{aligned}$$

But $1 - \cos^2\theta = \sin^2\theta$.

$$= \frac{\sin\theta(1-\cos\theta)}{\sin^2\theta}$$

Cancel.

$$= \frac{1 - \cos\theta}{\sin\theta}$$

□

Exercise 3(c) Proof: Use the sine expansion.

$$\begin{aligned}\sin 3\theta &= \sin(2\theta + \theta)\\ &= \sin 2\theta \cos\theta + \sin\theta \cos 2\theta\end{aligned}$$

But $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = 1 - 2\sin^2\theta$.

$$\begin{aligned}&= 2\sin\theta\cos\theta\cos\theta + \sin\theta(1 - 2\sin^2\theta)\\ &= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta\end{aligned}$$

But $\cos^2\theta = 1 - \sin^2\theta$.

$$\begin{aligned}&= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta\\ &= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta\\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

□

Exercise 4(a) Factor.

$$\begin{aligned}2\sin^2 x - 3\sin x + 1 &= 0\\ (2\sin x - 1)(\sin x - 1) &= 0\end{aligned}$$

Set each factor equal to zero and solve.

$$\begin{aligned}2\sin x - 1 &= 0 &\quad \text{or} \quad \sin x - 1 &= 0\\ 2\sin x &= 1 & \sin x &= 1\\ \sin x &= \frac{1}{2} & x &= \frac{\pi}{2}\\ x &= \frac{\pi}{6}, \frac{5\pi}{6} & &\end{aligned}$$

□

Exercise 4(b) Take teh square roots (plus or minus).

$$\begin{aligned}\tan^2 2x &= 3\\ \tan 2x &= \pm 3\end{aligned}$$

The tangent is equal to $\pm\sqrt{4}$ at 4 places on the unit circle. But we are seeking the tangent of $2x$, so we make two trips around the unit circle.

$$2x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}.$$

Multiply by 1/2.

$$x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{6}.$$

□

Exercise 4(c) We know that $\sin 2x = 2\sin x\cos x$.

$$\begin{aligned}\sin 2x &= \cos x\\ 2\sin x\cos x &= \cos x\end{aligned}$$

Make one side zero and factor.

$$\begin{aligned}2\sin x\cos x - \cos x &= 0\\ \cos x(2\sin x - 1) &= 0\end{aligned}$$

Set each factor equal to zero and solve.

$$\begin{aligned} \cos x &= 0 & \quad\text{or}\quad 2\sin x - 1 &= 0 \\ x &= \frac{\pi}{2}, \frac{3\pi}{2} & 2\sin x &= 1 \\ & & \sin x &= \frac{1}{2} \\ & & x &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

□

Exercise 5(a) The period is π. Thus,

$$\begin{aligned} T &= \frac{2\pi}{B} \\ \pi &= \frac{2\pi}{B} \\ B\pi &= 2\pi \\ B &= 2 \end{aligned}$$

The amplitude is 2. I see an inverted cosine with no phase shift (there are other choices). Thus,

$$y = -2\cos 2x.$$

□

Exercise 5(b) The period is $\pi/3$. Thus,

$$\begin{aligned} T &= \frac{2\pi}{B} \\ \frac{\pi}{3} &= \frac{2\pi}{B} \\ B\pi &= 6\pi \\ B &= 6 \end{aligned}$$

The amplitude is 3. I see a sine curve with no phase shift (there are other choices). Thus,

$$y = 3\sin 6x.$$

□

Exercise 5(c) The period is π. Thus,

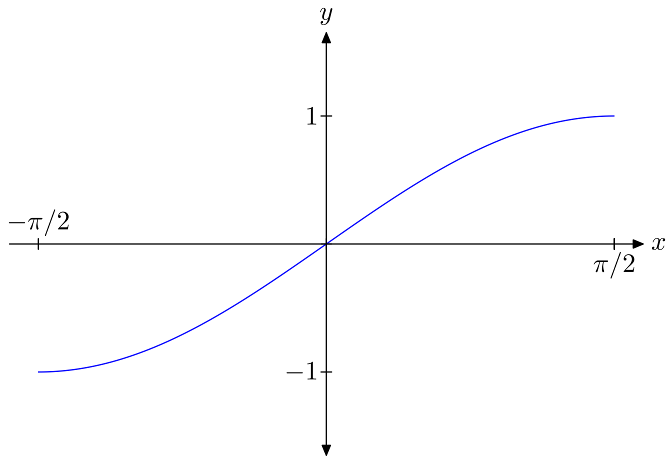
$$\begin{aligned} T &= \frac{2\pi}{B} \\ \pi &= \frac{2\pi}{B} \\ B\pi &= 2\pi \\ B &= 2 \end{aligned}$$

I see an inverted secant ($\sec x = 1/\cos x$). Thus,

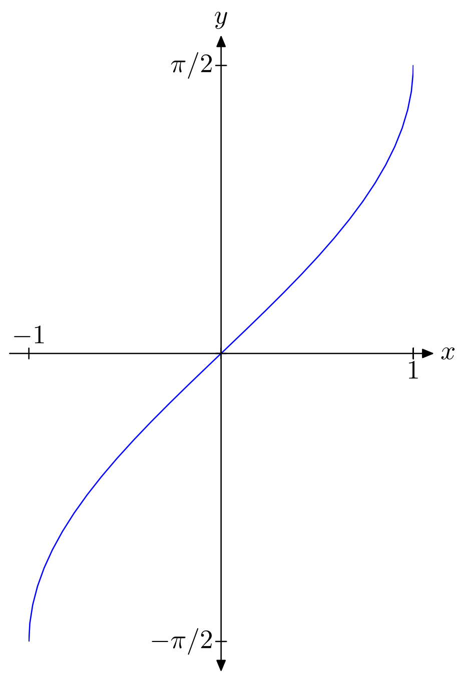
$$y = -\sec 2x.$$

□

Exercise 6(a) Select a portion of the sine curve that is one-to-one (passes the horizontal line test).



Reflect this piece across the line $y = x$.



The domain is $D = [-1, 1]$. The range is $R = [-\pi/2, \pi/2]$. □

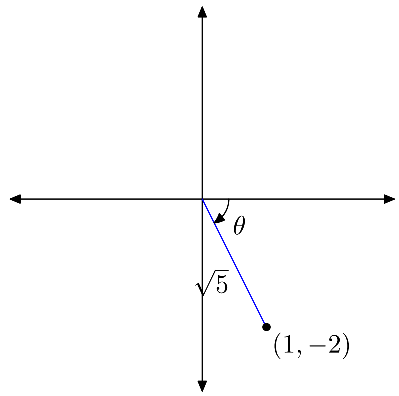
Exercise 6(b) Let

$$\cos(\tan^{-1}(-2)) = \cos\theta,$$

where $\theta = \tan^{-1}(-2)$. Therefore,

$$\tan\theta = -2.$$

θ is an angle in the fourth quadrant ($\tan^{-1}$ returns an angle $-\pi/2 < \theta < \pi/2$ and $\tan\theta$ is negative) and we can draw

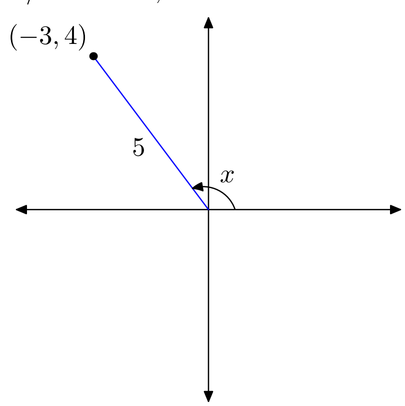


Thus,

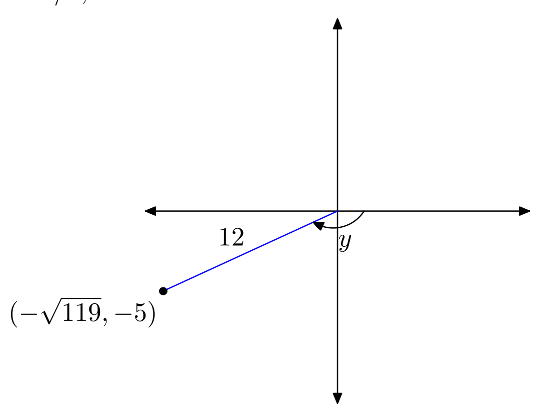
$$\begin{aligned}\cos(\tan^{-1}(-2)) &= \cos\theta \\ &= \frac{1}{\sqrt{5}} \\ &= \frac{\sqrt{5}}{5}\end{aligned}$$

□

Exercise 7(a) Because $\sin x = 4/5$, $\pi/2 < x < \pi$, we can draw



Because $\sin y = -5/12$, $\pi < y < 3\pi/2$, we can draw

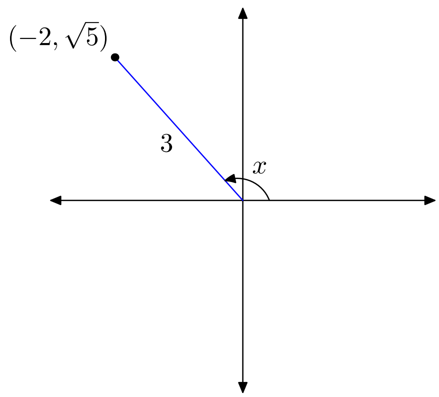


Thus,

$$\begin{aligned}\sin(x+y) &= \sin x\cos y + \sin y \cos x\\ &= \left(\frac{4}{5}\right)\left(-\frac{\sqrt{119}}{12}\right) + \left(-\frac{5}{12}\right)\left(-\frac{3}{5}\right)\\ &= -\frac{4\sqrt{119}}{60} + \frac{15}{60}\\ &= \frac{15-4\sqrt{119}}{60}.\end{aligned}$$

□

Exercise 7(b) Because $\cos x = -2/3$, $\pi/2 < x < \pi$, we can draw



Now, $\cos 2x = \cos^2 x - \sin^2 x$ (there are other choices), so

$$\begin{aligned}\cos 2x &= \left(-\frac{2}{3}\right)^2 - \left(\frac{\sqrt{5}}{3}\right)^2\\ &= \frac{4}{9} - \frac{5}{9}\\ &= -\frac{1}{9}\end{aligned}$$

□

Exercise 7(c) We can start with the half angle identity for sine. Note that $\pi/8$ is in the first quadrant where the sine is positive. Thus,

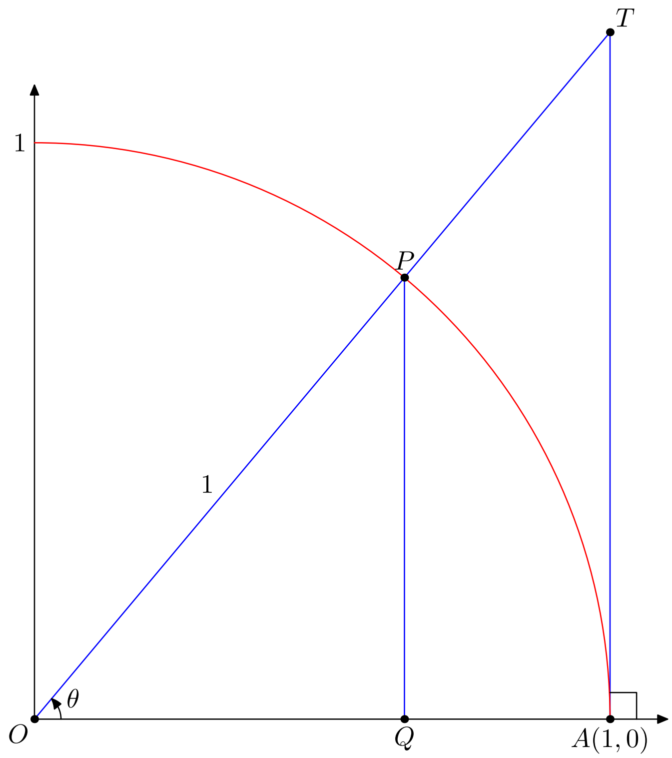
$$\begin{aligned}\sin\frac{\pi}{8} &= \sqrt{\frac{1-\cos\frac{\pi}{4}}{2}}\\ &= \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}.\end{aligned}$$

Multiply numerator and denominator by 2, then take the square root of both.

$$\begin{aligned}&= \sqrt{\frac{2-\sqrt{2}}{4}}\\ &= \frac{\sqrt{2-\sqrt{2}}}{2}\end{aligned}$$

□

Exercise 8(a) Note that the radius OP has length 1.



Triangle $\triangle OQP$ is a right triangle. Thus,

$$\cos\theta = \frac{\text{adj}}{\text{hyp}}$$
$$\cos\theta = \frac{OQ}{1}$$
$$\cos\theta = OQ$$

Therefore, $OQ = \cos\theta$. Next,

$$\sin\theta = \frac{\text{opp}}{\text{adj}}$$
$$\sin\theta = \frac{QP}{1}$$
$$\sin\theta = QP$$

Thus, $QP = \sin\theta$. □

Exercise 8(b) The formula relating the central angle, radius, and subtended arc is

$$\theta = \frac{s}{r}$$
$$s = r\theta.$$

But $s = \widehat{AP}$, and $r = 1$, so

$$\widehat{AP} = \theta.$$

□

Exercise 8(c) Use the tangent on this one.

$$\tan\theta = \frac{\text{opp}}{\text{adj}}$$
$$\tan\theta = \frac{AT}{OA}$$

But $OA = 1$, so

$$\tan\theta = \frac{AT}{1}$$
$$\tan\theta = AT$$

Therefore, $AT = \tan\theta$. □

Exercise 9(a) The diameter is 60 feet, so the radius is 30 feet. The angular velocity is 1 rev/s.

$$\omega = 1\,\text{rev/s}$$

But there are 2π radians in every revolution, so

$$\omega = \frac{1}{20}\frac{\text{rev}}{\text{s}} \times 2\pi\frac{\text{rad}}{\text{rev}}$$
$$\omega = \frac{\pi}{10}\frac{\text{rad}}{\text{s}}$$

□

Exercise 9(b) The formula relating speed, radius, and angular velocity is $v = \omega r$. Thus,

$$v = \omega r$$
$$v = \left(\frac{\pi}{10}\frac{\text{rad}}{\text{s}}\right)(3\,\text{ft})$$
$$v = 3\pi\,\text{ft/s}$$

□