Humboldt State University
Mathematics Department
Math 115—Precalculus

Exam #3—Trigonometry

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Essay Questions

Instructions: *Place the solution to each of the following problems on a separate sheet of paper (you may use the front and back sides). It is required that you do 7 of the following 9 questions, no more, no less. When you complete the exam, arrange your solutions in order, then staple the exam on top as a cover page. Place your name on the exam. You will be docked severely for not following instructions. For example, if you put two or more problems on one sheet of paper, there will be a deduction. If you hand in 8 problems instead of 7, then I will correct 7 of my choice. Rest assured that one of my choices will be the one you least wanted graded. If your solutions are not arranged in order, there will be a deduction. If your name is not on the cover sheet, there will be a deduction, etc., etc. Follow directions or lose points! With that said, good luck!*

Exercise 1. Mark Twain sat on the deck of a river steamboat. As the paddlewheel turned, a point on the paddle blade moved in such a way that its distance, d, from the water's surface was a sinusoidal function of time. When his stopwatch read 4 seconds, the point was at its highest, 16 feet above the water's surface. The wheel's diameter was 18 feet, and it completed a revolution every 10 seconds.

(a) Sketch a graph of the sinusoid.

(b) Write an equation of the sinusoid. That is, express d in terms of t, the amount of time that has passed since Mark Twain started his stopwatch.

(c) How far above the surface was the point on the paddlewheel when Mark's stopwatch read 5 seconds?

Exercise 2. Researchers find a creature from an alien planet. Its body temperature is varying sinusoidally with time. 35 minutes after they start timing, it reaches a high of $120°$ F. 20 minutes after that (55 minutes after they started timing) it reaches its next low, $104°$ F.

(a) Sketch a graph of this function.

(b) Write an equation expressing temperature in terms of minutes since they started timing.

(c) What was its temperature when they first started timing?

Exercise 3. Prove any two of the following three identities.

(a) $\sec\theta + \tan\theta = \dfrac{1}{\sec\theta - \tan\theta}$

(b) $\dfrac{\sin 2\theta}{1 + \cos 2\theta} = \tan\theta$

(c) $\sin x \sin y = \frac{1}{2}\left[\,\cos(x-y) - \cos(x+y)\right]$

Exercise 4. Solve any two of the following three equations on the interval $[0, 2\pi)$. I am not interested in solutions found by a calculator. Rather, I want to see all appropriate work: factoring, identities, etc. Also, please give exact answers only. Decimal approximations will receive no credit.

(a) $\cos 2x = \sin x$

(b) $2\cos 5x \cos 3x + 2\sin 5x \sin 3x = 1$

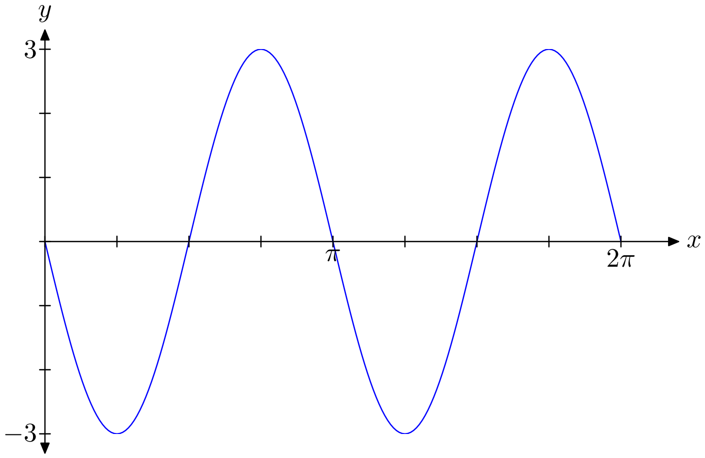
(c) $4\sin x \cos x = \sqrt{3}$

Exercise 5. Answer each of the following questions.

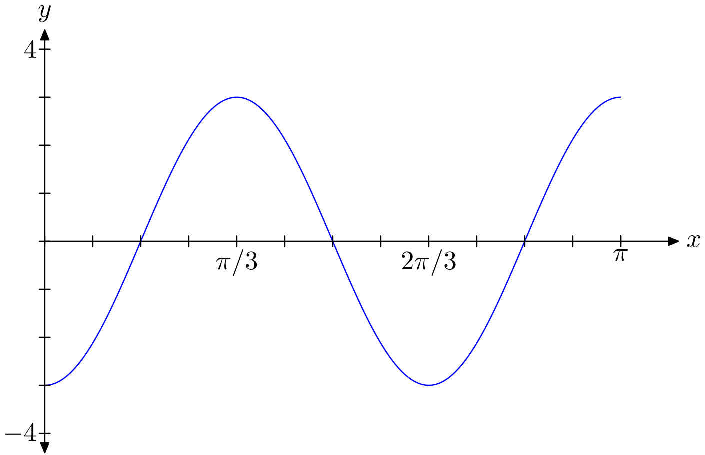
(a) Sketch the graph of $y = \cos^{-1} x$. State the domain and range using interval notation. No decimal approximations will be accepted.

(b) Evaluate $\sec(\tan^{-1}(-3))$. Show all appropriate work and place answer in simple radical form. No credit will be given for any decimal approximations or answers without supportive work.

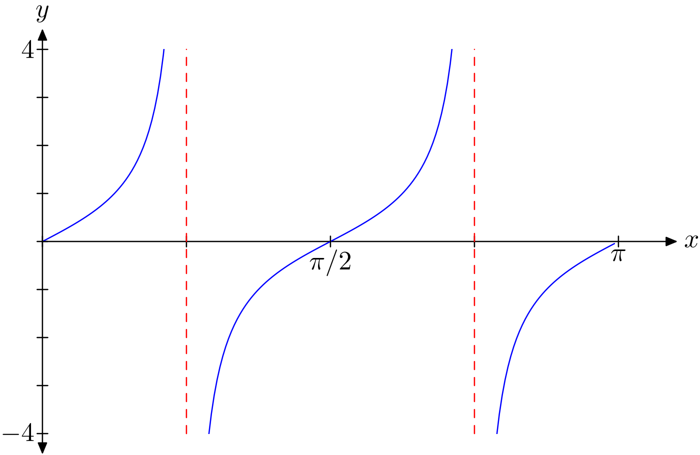
Exercise 6. Answer any two of the following three questions. In each case, state the amplitude, period, and phase shift (if any), then give the equation of the given graph.



(a)



(b)



(c)

Exercise 7. Answer any two of the following three questions. No credit will be given for any decimal approximations or solutions without supportive work. Place all answers in simple radical form.

(a) If $\tan x = 4$, $\pi < x < 3\pi/2$, and $\tan y = -5$, $\pi/2 < y < \pi$, find the exact value of $\tan(x + y)$.

(b) Find the exact value of $\sin \dfrac{\pi}{8}$.

(c) Find an exact value of $\cos 75^\circ$ in simple radical form.

Exercise 8. Answer any two of the following three questions.

(a) Write $\sin(\arccos x)$ as an algebraic expression involving x.

(b) Write $\sin(\arcsin x + \arccos x)$ as an algebraic expression involving x.

(c) Write $\sin(2 \arccos x)$ as an algebraic expression involving x.

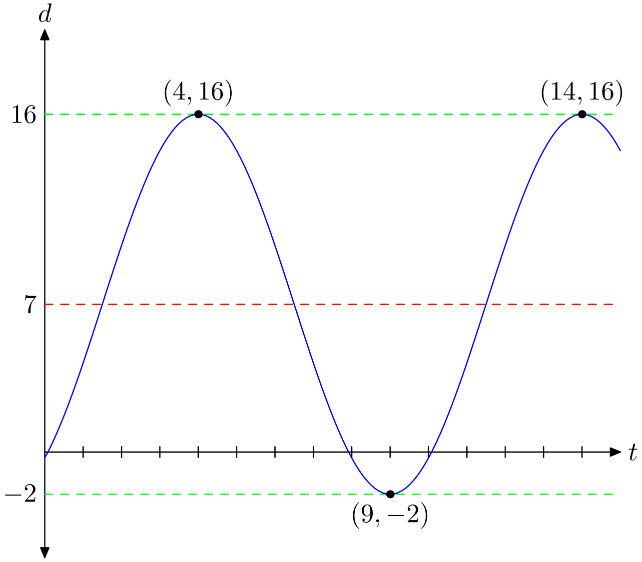
Exercise 9. A car is moving at a rate of 50 miles per hour and the *diameter* of its wheels is 2.5 feet.

(a) Find the angular speed of its wheels in radians per minute (5280 feet equals 1 mile).

(b) Find the angular speed of its wheels in revolutions per minute.

Solutions to Exercises

Exercise 1(a) Because the wheel makes 1 revolution every 10 seconds, the point on the wheel will return to its original position every 10 seconds. Hence, the period is 10 seconds. At $t = 4$ seconds, the point on the wheel is at its highest point, 16 feet above the surface of the water. Because the wheel is 18 feet in diameter, the point on the wheel will be 2 feet below the surface 5 seconds later, at $t = 9$ seconds. Five seconds after that, at $t = 14$ seconds, the point will be at its highest position once again.


d
(4, 16)
(14, 16)
16
7
t
−2
(9, −2)

□

Exercise 1(b) Because the period is 10 seconds,

$$\begin{aligned} T &= \frac{2\pi}{B} \\ 10 &= \frac{2\pi}{B} \\ 10B &= 2\pi \\ B &= \frac{2\pi}{10} \\ B &= \frac{\pi}{5}. \end{aligned}$$

The amplitude is found by taking half the difference of the maximum and minimum heights.

$$\text{Amp} = \frac{16-(-2)}{2} = 9$$

I will use a cosine curve that is shifted 4 seconds to the right and 7 feet up.

$$d = 9\cos\frac{\pi}{5}(t-4)+7$$

□

Exercise 1(c) At $t = 5$ seconds,

$$d = 9\cos\frac{\pi}{5}(5-4) + 7$$
$$d \approx 14.3 \text{ feet.}$$

□

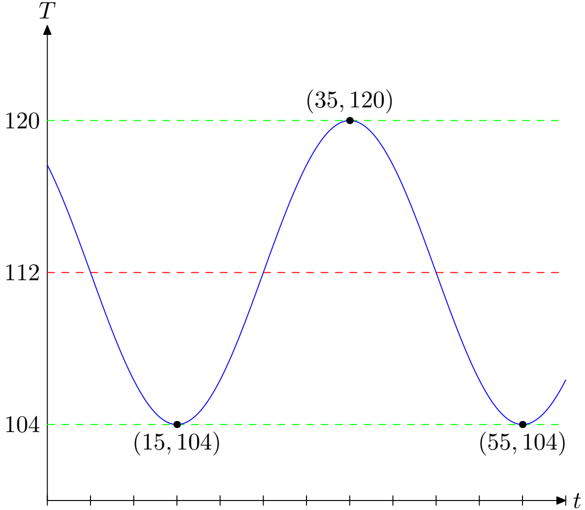
Exercise 2(a) At $t = 35$ minutes, the temperature is at a maximum of 120° F. Twenty minutes later, at $t = 55$ minutes, the temperature is at a minimum of 104° F. Thus, half of the period is determined by

$$\frac{1}{2}T = 55 - 35$$
$$\frac{1}{2}T = 20\,\text{min}$$

Therefore,

$$T = 40\,\text{min}$$

is the period.


T
120
112
104
(35, 120)
(15, 104)
(55, 104)
t

□

Exercise 2(b) Since the period is 40 minutes,

$$\begin{aligned} T &= \frac{2\pi}{B} \\ 40 &= \frac{2\pi}{B} \\ 40B &= 2\pi \\ B &= \frac{2\pi}{40} \\ B &= \frac{\pi}{20}. \end{aligned}$$

The amplitude is

$$\text{Amplitude} = \frac{120 - 104}{2} = \frac{16}{2} = 8.$$

I will work with the cosine that begins at $(35, 120)$, which is a shift of 35 minutes to the right and a shift upward of 112° F. Thus,

$$T = 8\cos\frac{\pi}{20}(t - 35) + 112.$$

□

Exercise 2(c) When timing began, $t = 0$, so

$$T = 8\cos\frac{\pi}{20}(0-35)+112$$
$$T \approx 117.7^\circ .$$

□

Exercise 3(a) *Proof*:

$$\begin{aligned}\sec\theta + \tan\theta &= \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} \\ &= \frac{1+\sin\theta}{\cos\theta}\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{1}{\sec\theta - \tan\theta} &= \frac{1}{\dfrac{1}{\cos\theta} - \dfrac{\sin\theta}{\cos\theta}} \\ &= \frac{1}{\dfrac{1-\sin\theta}{\cos\theta}} \\ &= \frac{\cos\theta}{1-\sin\theta}.\end{aligned}$$

Multiply numerator and denominator by $1 + \sin\theta$.

$$\begin{aligned}
&= \frac{(\cos\theta)(1+\sin\theta)}{(1-\sin\theta)(1+\sin\theta)} \\
&= \frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta} \\
&= \frac{\cos\theta(1+\sin\theta)}{\cos^2\theta} \\
&= \frac{1+\sin\theta}{\cos\theta}
\end{aligned}$$

Therefore,

$$\sec\theta + \tan\theta = \frac{1}{\sec\theta - \tan\theta}.$$

□

Exercise 3(b) *Proof*:

$$\begin{aligned}
\frac{\sin 2\theta}{1+\cos 2\theta} &= \frac{2\sin\theta\cos\theta}{1+(2\cos^2\theta-1)} \\
&= \frac{2\sin\theta\cos\theta}{2\cos^2\theta} \\
&= \frac{\sin\theta}{\cos\theta} \\
&= \tan\theta.
\end{aligned}$$

□

Exercise 3(c) *Proof*:

$$\begin{aligned}
&\frac{1}{2}[\cos(x-y) - \cos(x+y)] \\
&\quad = \frac{1}{2}[(\cos x \cos y + \sin x \sin y) - (\cos x \cos y - \sin x \sin y)] \\
&\quad = \frac{1}{2}[2 \sin x \sin y] \\
&\quad = \sin x \sin y
\end{aligned}$$

□

Exercise 4(a) Since $\cos 2x = 1 - 2\sin^2 x$, we can write

$$\begin{aligned} \cos 2x &= \sin x \\ 1 - 2\sin^2 x &= \sin x \\ 0 &= 2\sin^2 x + \sin x - 1. \end{aligned}$$

Factor.

$$0 = (2\sin x - 1)(\sin x + 1)$$

Set each factor equal to zero.

$$\begin{aligned} 2\sin x - 1 &= 0 \\ \sin x &= \frac{1}{2} \\ x &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned} \qquad \begin{aligned} \sin x + 1 &= 0 \\ \sin x &= -1 \\ x &= \frac{3\pi}{2} \end{aligned}$$

□

Exercise 4(b) Use the cosine expansion to "collapse" the expression on the left.

$$\begin{aligned} 2\cos 5x\cos 3x + 2\sin 5x\sin 3x &= 1 \\ \cos 5x\cos 3x + \sin 5x\sin 3x &= \frac{1}{2} \\ \cos(5x - 3x) &= \frac{1}{2} \\ \cos 2x &= \frac{1}{2} \end{aligned}$$

Thus,

$$\begin{aligned} 2x &= \frac{\pi}{3},\ \frac{5\pi}{3},\ \frac{7\pi}{3},\ \frac{11\pi}{3} \\ x &= \frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{7\pi}{6},\ \frac{11\pi}{6} \end{aligned}$$

□

Exercise 4(c) Since $\sin 2x = 2\sin x \cos x$, we can write

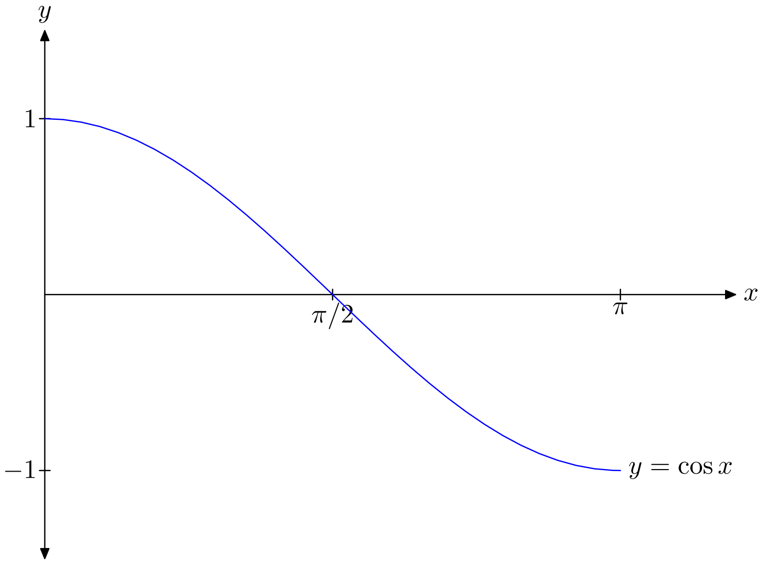
$$\begin{aligned} 4\sin x\cos x &= \sqrt{3} \\ 2\sin x\cos x &= \frac{\sqrt{3}}{2} \\ \sin 2x &= \frac{\sqrt{3}}{2}. \end{aligned}$$

Thus,

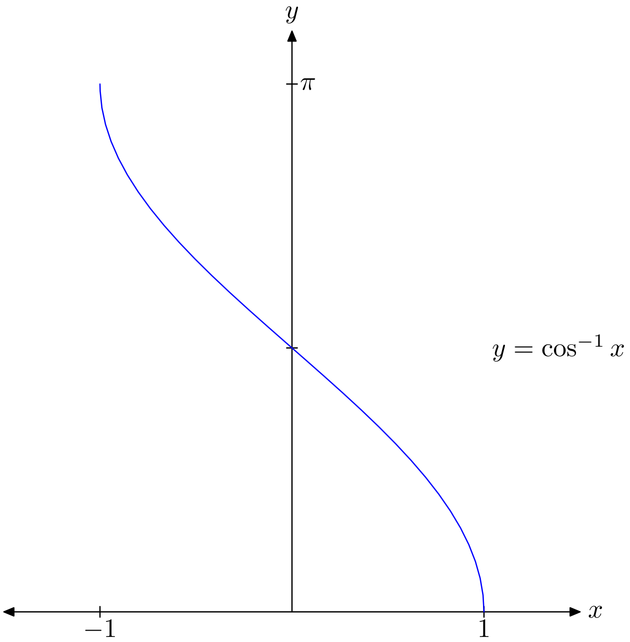
$$\begin{aligned} 2x &= \frac{\pi}{2}, \ \frac{2\pi}{3}, \ \frac{7\pi}{3}, \ \frac{8\pi}{3} \\ x &= \frac{\pi}{6}, \ \frac{\pi}{3}, \ \frac{7\pi}{6}, \ \frac{4\pi}{3} \end{aligned}$$

□

Exercise 5(a) Restrict the cosine so that it passes the horizontal line test and is one-to-one.



Reflect this graph across the line $y = x$.


y
π
$y = \cos^{-1} x$
x
−1
1

Project onto the x-axis to find the domain.

$$D = [-1, 1].$$

Project onto the y-axis to find the range.

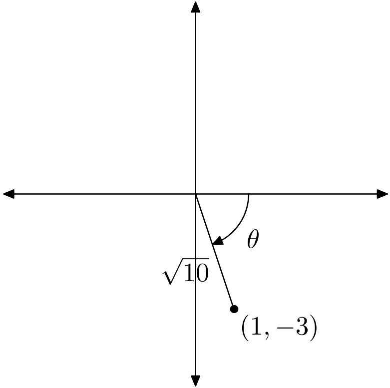
$$R = [0, \pi]$$

□

Exercise 5(b) Let

$$\sec(\tan^{-1}(-3)) = \sec\theta, \tag{1}$$

where $\theta = \tan^{-1}(-3)$. Thus, $\tan\theta = -3$ and θ must locate in the fourth quadrant.



Because $\tan\theta = y/x$, choose the endpoint of the radius to be the point $(1, -3)$. Thus insuring that

$$\tan\theta = \frac{y}{x} = \frac{-3}{1} = -3.$$

Calculate the radius with

$$r = \sqrt{(1)^2 + (-3)^2} = \sqrt{1+9} = \sqrt{10}.$$

Now,

$$\cos\theta = \frac{x}{r}$$

$$\cos\theta = \frac{1}{\sqrt{10}},$$

so

$$\sec\theta = \sqrt{10}.$$

Substitute this result in (1) to get

$$\sec(\tan^{-1}(-3)) = \sqrt{10}.$$

□

Exercise 6(a) The curve has

$$\text{Amplitude} = \frac{3-(-3)}{2} = 3.$$

The period is π, so

$$\begin{aligned} T &= \frac{2\pi}{B} \\ \pi &= \frac{2\pi}{B} \\ B\pi &= 2\pi \\ B &= 2. \end{aligned}$$

The first period of the curve looks like an inverted sine, so

$$y = -3\sin 2x.$$

□

Exercise 6(b) The curve has

$$\text{Amplitude} = \frac{3-(-3)}{2} = 3.$$

The first minimum is at $(0, -3)$, the next at $(2\pi/3, -3)$. Thus, the period is $2\pi/3$. Moreover,

$$\begin{aligned} T &= \frac{2\pi}{B} \\ \frac{2\pi}{3} &= \frac{2\pi}{B} \\ B &= 3. \end{aligned}$$

The first period of the curve appears to be an inverted cosine. Thus,

$$y = -3\cos 3x.$$

□

Exercise 6(c) The first period occurs on $[0, \pi/2]$. Thus, the period is $\pi/2$ and

$$\begin{aligned} T &= \frac{\pi}{B} \\ \frac{\pi}{2} &= \frac{\pi}{B} \\ B &= 2. \end{aligned}$$

The graph is a simple tangent curve. Thus,

$$y = \tan 2x.$$

□

Exercise 7(a) If $\tan x = 4$ and $\tan y = -5$, then

$$\begin{aligned} \tan(x+y) &= \frac{\tan x + \tan y}{1 - \tan x \tan y} \\ &= \frac{4 + (-5)}{1 - (4)(-5)} \\ &= \frac{-1}{1+20} \\ &= -\frac{1}{21}. \end{aligned}$$

□

Exercise 7(b) Using the half angle identity for sine,

$$\begin{aligned}\sin\frac{\pi}{8} &= \sqrt{\frac{1-\cos\pi/4}{2}}\\ &= \sqrt{\frac{1-\dfrac{\sqrt{2}}{2}}{2}}\\ &= \sqrt{\frac{2-\sqrt{2}}{4}}\\ &= \frac{\sqrt{2-\sqrt{2}}}{2}.\end{aligned}$$

□

Exercise 7(c) Using the cosine expansion,

$$\begin{aligned}\cos 75^\circ &= \cos(45^\circ + 30^\circ)\\ &= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ\\ &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)\\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}.\end{aligned}$$

□

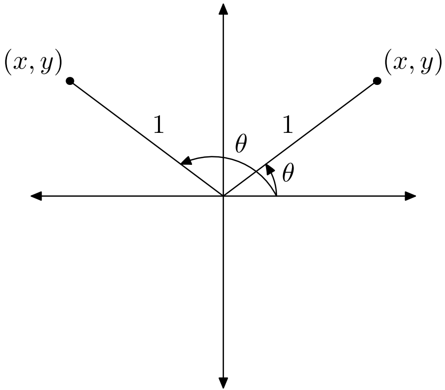
Exercise 8(a) Let

$$\sin(\arccos x) = \sin \theta, \tag{2}$$

where

$$\theta = \arccos x.$$

Because the range of the arccos is $[0, \pi]$, θ could lie in quadrant I or II.



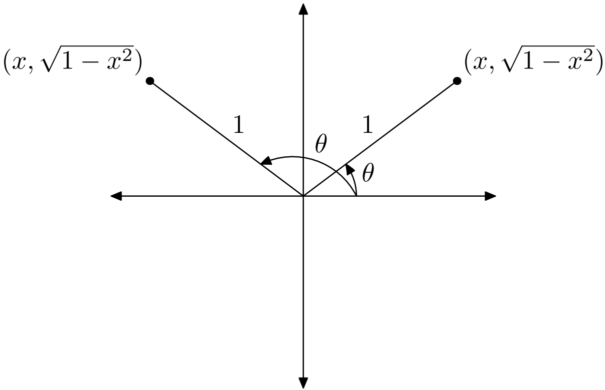
Because $\theta = \arccos x$, we know that $\cos\theta = x$. because the definition of cosine is $\cos\theta = x/r$, choose the point at the end of the radial vector to be (x, y) and the radial length to be 1. This insures that

$$\cos\theta = \frac{x}{r} = \frac{x}{1} = x.$$

Calculate y with

$$\begin{aligned} x^2 + y^2 &= r^2 \\ x^2 + y^2 &= 1^2 \\ y^2 &= 1 - x^2 \\ y &= \pm\sqrt{1 - x^2}. \end{aligned}$$

Because we are in quadrant I or II, choose $y = \sqrt{1 - x^2}$ and draw



Thus, in both cases,

$$\sin\theta = \frac{y}{r} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}.$$

Substitute this result in (2) to obtain

$$\sin(\arccos x) = \sin\theta$$
$$\sin(\arccos x) = \sqrt{1 - x^2}.$$

□

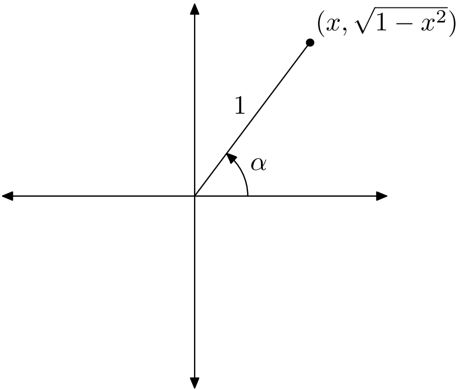
Exercise 8(b) Write

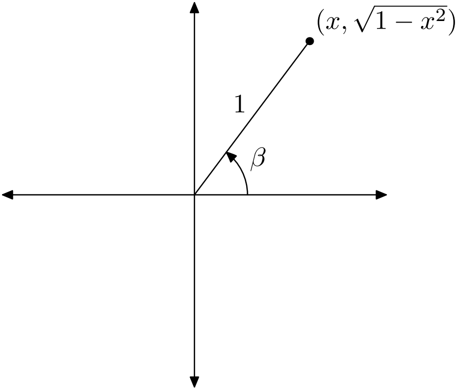
$$\sin(\arcsin x + \arccos x) = \sin(\alpha + \beta),$$

where

$$\begin{aligned} \alpha &= \arcsin x & \beta &= \arccos x \\ \sin\alpha &= x & \cos\beta &= x. \end{aligned}$$

Using the technique from Exercise 8a,


$(x, \sqrt{1-x^2})$
1
α



The picture on the left shows that

$$\cos\alpha = \sqrt{1-x^2} \quad \text{and} \quad \sin\alpha = x.$$

The picture on the right shows that

$$\cos\beta = x \quad \text{and} \quad \sin\beta = \sqrt{1-x^2}.$$

Thus,

$$\begin{aligned}
\sin(\arcsin x + \arccos x) &= \sin(\alpha + \beta) \\
&= \sin \alpha \cos \beta + \sin \beta \cos \alpha \\
&= (x)(x) + (\sqrt{1 - x^2})(\sqrt{1 - x^2}) \\
&= x^2 + 1 - x^2 \\
&= 1.
\end{aligned}$$

□

Exercise 8(c) Write

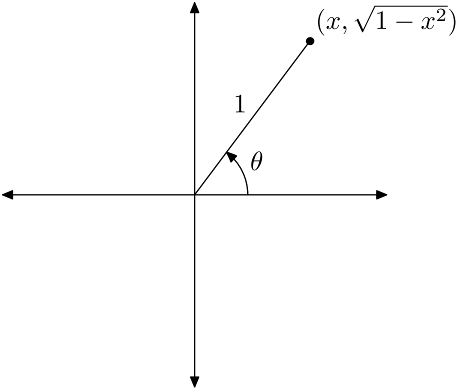
$$\sin(2\arccos x) = \sin 2\theta,$$

where

$$\theta = \arccos x$$

$$\cos\theta = x.$$

Using the technique in Exercises 8a and 8b, draw



Thus,

$$\cos\theta = x \quad \text{and} \quad \sin\theta = \sqrt{1-x^2}.$$

Therefore,

$$\begin{aligned}\sin(2\arccos x) &= \sin 2\theta \\ &= 2\sin\theta\cos\theta \\ &= 2(\sqrt{1-x^2})(x) \\ &= 2x\sqrt{1-x^2}.\end{aligned}$$

□

Exercise 9(a) A point on the wheel will cover the same distance as that travelled by the car, so the speed of a point on the rim o f the wheel is 50 miles per hour. We first change this speed into feet per minute with the calculation

$$v = \frac{50\,\text{mi}}{\text{h}} \times \frac{5280\,\text{ft}}{\text{mi}} \times \frac{1\,\text{h}}{60\,\text{min}}$$
$$v = 4400\,\text{ft/min}.$$

The angular speed is calculated with

$$v = \omega r$$
$$\omega = \frac{v}{r}$$
$$\omega = \frac{4400\,\text{ft/min}}{1.25\,\text{ft}}$$
$$\omega = 3520\,\text{rad/min}.$$

□

Exercise 9(b) To change to revolutions per minute, use the fact that the wheel turns through 2π radians every revolution.

$$\omega = \frac{3520\,\text{rad}}{\text{min}} \times \frac{1\,\text{rev}}{2\pi\,\text{rad}}$$
$$\omega = \frac{3520}{2\pi}\,\frac{\text{rev}}{\text{rev}}$$
$$\omega \approx 560\,\text{rev/min}$$

□