College of the Redwoods
Mathematics Department
Math 45—Linear Algebra

Pretest–Exam #4
Linear Algebra

David Arnold

Last Revision Date: December 1, 2000 Version 1.00

Essay Questions

Directions: *Place the solution to each of the following exercises on your own paper. You must follow directions explicitly and show all work to receive full credit.*

Exercise 1. Find

(a) the projection of the vector $(2, 1)$ onto the vector $(1, 1)$, and

(b) find the projection matrix P that will project all vectors in the plane onto the line through the origin that points in the direction $(1, 1)$,

Exercise 2. In 3-space,

(a) find the equation of the line passing through the origin in the direction of $(1, 1, 1)$, and

(b) find the equation of the plane passing through the origin that is perpendicular to the line in part (a).

Exercise 3. What is the shortest distance between the line $y = 2x+3$ and the point $(5, -2)$?

Exercise 4. Find the point on the line $y = 1 - 2x$ that is closest to the point $(3, 1)$.

Exercise 5. Find the distance between the point $(1, 2, -1)$ and the plane $x + 2y + 3z = 0$.

Exercise 6. Consider the plane in $\mathbb{R}^3$ spanned by the vectors $(1, 1, 0)^T$ and $(0, 1, 1)^T$.

(a) Find the projection matrix P that will project all vectors in $\mathbb{R}^3$ onto the plane defined above.

(b) Find the projection of the vector $\mathbf{b} = (1, 2, -3)^T$ onto the plane defined above.

Exercise 7. Consider the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find bases for the four fundamental subspaces. Place your results on a Strang diagram.

Exercise 8. Find the eigenvalues and eigenvectors of each matrix.

(a)

$$\begin{bmatrix} 7 & -10 \\ 5 & -8 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 5 & -3 & 4 \\ 0 & 2 & 0 \\ -8 & 8 & -7 \end{bmatrix}$$

Exercise 9. Consider the following table of data points.

x	y
0	5
1	1
2	1

(a) Plot the data on graph paper.

(b) Find the equation of the line of best fit (hand calculations only) without the aid of technology.

(c) Draw the line of best fit on your graph paper,

(d) Find the sum of the squares of the errors made.

Exercise 10. Prove that similar matrices have the same eigenvalues.

Exercise 11. Let R be a 2×2 matrix that reflects vectors across a given line in the plane. Use the geometry of the situation to find the eigenvalues of the matrix.

Exercise 12. Consider the matrix

$$\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$$

(a) Find the 4 fundamental subspaces.

(b) Draw $N(A)$ and $C(A^T)$ in the plane.

(c) Draw $C(A)$ and $N(A^T)$ in the plane.

Exercise 13. If we remember our theory of linear transformations, we know that any transformation $P : \mathbb{R}^2 \to \mathbb{R}^2$ is completely determined by its action on the "standard basis" vectors. Let P be the matrix that projects vectors in the plane onto the line spanned by the vector $(-1, 2)^T$.

(a) The formula for projecting vector $\mathbf{b}$ onto $\mathbf{a}$ is

$$\mathbf{p} = \frac{\mathbf{b} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}} \mathbf{a}$$

Form the matrix P, where the first column of P is the projection of $\mathbf{e}_1 = (1, 0)^T$ onto $\mathbf{a} = (-1, 2)$ and the second column of P is the projection of $\mathbf{e}_2 = (0, 1)^T$ onto $\mathbf{a} = (-1, 2)^T$.

(b) Use the formula

$$P = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}$$

to compute the matrix P that projects vectors in the plane onto the line spanned by $\mathbf{a} = (-1, 2)^T$. Compare this with the result found in part (a).

Exercise 14. Find a matrix A that has

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

as a basis for its column space and

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

as a basis for its row space ($C(A^T)$).

Exercise 15. Find a matrix whose row space is spanned by $(1, 1, 2)^T$ and whose null space is spanned by

(a) $(1, 2, -1)^T$.

(b) $(1, 2, -1)^T$.

Exercise 16. If A is a 7×9 matrix with nullity 2, what are the dimensions of the 4 fundamental subspaces? Include a Strang diagram.

Here are a few fun questions from Professor Strang's MIT quizzes.

Exercise 17. Suppose

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 7 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 4 & 5 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

(a) What is the rank of A?

(b) Find a basis for the nullspace of A.

(c) Find the complete solution to

$$A\mathbf{x} = \begin{bmatrix} 10 \\ 15 \\ 85 \end{bmatrix}$$

Exercise 18. Suppose that row operations (elimination) reduce the matrices A and B to the same row echelon form

$$R = \begin{bmatrix} 1 & 2 & 0 & 7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Which of the four subspaces are sure to be the same for A and B? ($C(A) = C(B)$? $N(A) = N(B)$? $C(A^T) = C(B^T)$? $N(A^T) = N(B^T)$?)

(b) Each time the subspaces in part (a) are the same for A and B, find a basis for that subspace.

(c) True or False (A is any matrix and $\mathbf{x}$, $\mathbf{y}$ are two vectors): If $A\mathbf{x}$ and $A\mathbf{y}$ are linearly independent then $\mathbf{x}$ and $\mathbf{y}$ are linearly independent.

EXERCISE 19. Suppose A is an m by n of rank r.

(a) If $A\mathbf{x} = \mathbf{b}$ has a solution for ever right side $\mathbf{b}$, what is the column space of A?

(b) In part (a), what are all equations or inequalities that must hold between the numbers m, n, and r.

(c) Give a specific example of a 3 by 2 matrix A of rank 1 with first row $[2\ 5]$. Describe the column space $C(A)$ and the nullspace $N(A)$ completely.

(d) Suppose the right side $\mathbf{b}$ is the same as the first column in your example (part c). Find the complete solution to $A\mathbf{x} = \mathbf{b}$.

Exercise 20. Let A by an $n \times n$ matrix.

(a) If the row space of A is $\mathbb{R}^n$ then the column space of A is ?

(b) If the nullspace of A is $\mathbb{R}^n$ then the column space of A is ?

(c) If the left nullspace of A is $\mathbb{R}^n$ then the column space of A is ?

(d) Give an example of a square matrix A such that the column space is orthogonal to the row space.

(e) If the column space of an $n \times n$ matrix A is orthogonal to the row space there is an inequality relating the rank r to n. What is the strongest possible inequality? (Hint: $r \leq n$ is a true inequality, but is not the strongest and hence will be considered an incorrect answer. Only the right answer will be given credit.)

(f) If the column space is orthogonal to the row space, then $\det(A) =$?

Solutions to Exercises

Exercise 1(a) □

Exercise 1(b) □

Exercise 2(a) □

Exercise 2(b) □

Exercise 3.

Exercise 3

Exercise 4. Exercise 4

Exercise 5. Exercise 5

Exercise 6(a) □

Exercise 6(b) □

Exercise 7. Exercise 7

Exercise 8(a) □

Exercise 8(b) □

Exercise 9(a) □

Exercise 9(b) □

Exercise 9(c) □

Exercise 9(d) □

Exercise 10.

Exercise 10

Exercise 11. Exercise 11

Exercise 12(a) □

Exercise 12(b) □

Exercise 12(c) □

Exercise 13(a) □

Exercise 13(b) □

Exercise 14.

Exercise 14

Exercise 15(a) □

Exercise 15(b) □

Exercise 16.

Exercise 16

Exercise 17(a) □

Exercise 17(b) □

Exercise 17(c) □

Exercise 18(a) □

Exercise 18(b) □

Exercise 18(c) □

Exercise 19(a) □

Exercise 19(b) □

Exercise 19(c) □

Exercise 19(d) □

Exercise 20(a) □

Exercise 20(b) □

Exercise 20(c) □

Exercise 20(d) □

Exercise 20(e) □

Exercise 20(f) □