Exam #3 — Sample Questions

Math 45 — Linear Algebra
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Instructions. *Place the solution to each of the following questions on your own paper. Show all of your work.*

1. Complete the following definitions:

a. Let H be a subset of a vector space V. Then H is a **subspace** of V if and only if …

b. Let $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ be a collection of elements of a vector space V. The **span** of $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ is …

c. Let $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ be a collection of elements of a vector space V. The vectors $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ are **linearly independent** if and only if …

d. Let $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ be a collection of elements of a vector space V. The vectors $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ are **linearly dependent** if and only if …

e. Let A be an $m \times n$ matrix. The **null space** of the matrix A is …

f. Let A be an $m \times n$ matrix. The **column space** of the matrix A is …

g. A transformation T mapping vector space V into vector space W is called a **linear transformation** if and only if …

h. Let H be a subspace of a vector space V. An set of vectors $B = \{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ is called a **basis** for the subspace H if and only if …

i. Let $B = \{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ be a basis for the vector space V and let $\mathbf{x}$ be a vector in V. The **coordinates of x relative to the basis** B are …

j. A transformation mapping vector space V into vector space W is called an **isomorphism** if and only if …

k. The **dimension** of a vector space V is …

l. Let A be an $m \times n$ matrix. The **nullity** of A is …

m. Let A be an $m \times n$ matrix. The **rank** of A is …

2. Let A be an $m \times n$ matrix. Prove that the null space of A is a subspace of R^n.

3. Let

$$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \le 4 \right\}$$

Show that H is **not** a subspace of R^2.

4. Let $M_{2\times 2}$ be the set of all 2×2 matrices with real entries. You may assume that $M_{2\times 2}$ is a vector space over the real numbers. Define a subset H of $M_{2\times 2}$ by

$$H = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a,b \in R \right\}$$

where R is the set of real numbers. Prove that H is a subspace of $M_{2\times 2}$.

5. Let

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ -1 & 2 & 2 & 0 \\ 0 & 0 & 3 & 5 \end{bmatrix}$$

Complete each of the following statements.

a. The null space of A is a subspace of ______.

b. The column space of A is a subspace of _____.

6. Let

$$A = \begin{bmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

a. Find a basis for the column space of A.

b. Find a basis for the null space of A.

7. Let

$$H = \left\{ \begin{bmatrix} 2a - 3b + d \\ 3a - 4c + 3e \\ c + 2d - e \end{bmatrix} : a, b, c, d, e \in R \right\}$$

Find a basis for H.

8. It can be shown that

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \right\}$$

is a basis for R^3.

a. If

$$[\mathbf{x}]_B = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

find **x**.

b. If

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

find $[\mathbf{x}]_B$.

9. Let V be a vector space with basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Define $T : V \to R^3$ by

$$T(x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

a. Prove that T is a linear transformation.

b. Prove that T is one-to-one.

c. Prove that T is onto R^3.

10. Let P_2 be the set of all polynomials of degree less than or equal to two with real coefficients; that is,

$$P_2 = \{c_0 + c_1 t + c_2 t^2 : c_1, c_2, c_3 \in R\}$$

You may assume that the set

$$B = \{1 + t^2, 3 - t, 1 + t + t^2\}$$

is a basis for P_2. If $p(t) = 3 + 2t + t^2$, find $[p(t)]_B$.

11. Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 & - \end{bmatrix}$$

a. Find the nullity of A.

b. Find the rank of A.