College of the Redwoods
Mathematics Department

Math 50A — Differential Calculus
Quiz #2

David Arnold

Last Revision Date: September 18, 2007 Version 1.00

Quiz Questions

Read Carefully! *You have until Monday of next week (9/17/07) to complete the quiz. The quiz is due at the beginning of class on Monday (9/17/07). Distance students should fax their quiz to 707-476-4424. Please see the guidelines for faxing homework and quizzes on the syllabus. Late quizzes are not accepted.*

This quiz is open notes, open book. This includes any supplementary texts or online documents. You must answer all of the exercises on your own. You are not allowed to work in groups or pairs on the quiz. You are not allowed to enlist the aid of a tutor or friend to help with the quiz. You are not allowed to read the exercises in the quiz, then seek help on similar questions. Once you open the quiz and read the questions, you may not seek any outside help of any kind.

I am not interested in reading pages and pages of calculations without accompanying narrative. It is essential that you include sound mathematical writing that both explains and justifies your solution or proof. Grammar and punctuation are important, as is the organization of your solution on the written page.

Place the solution to each exercise on a separate *sheet of paper. On a good sheet of paper, write out (longhand) and sign the following honor pledge.*

> I promise that all work found herein is my own. I have received no help from tutors, colleagues, or other teachers. I also promise that I have refrained from sharing my work and ideas with other students in the class. I have also honored all of the quiz constraints listed in the directions.

Arrange your solutions in order, place these quiz page(s) on top of your solutions, then place the honor pledge on top of the quiz as a cover sheet. Staple. Good luck!

Exercise 1. Consider the function

$$f(x) = \frac{x^2 - 2x - 3}{|x - 3|}. \tag{1}$$

(a) Construct a piecewise definition for the function f and use the result to evaluate each of the following limits:

1. $\lim_{x \to 3^-} f(x)$, and
2. $\lim_{x \to 3^+} f(x)$.

Solution: First, factor the numerator of f.

$$f(x) = \frac{x^2 - 2x - 3}{|x - 3|} = \frac{(x - 3)(x + 1)}{|x - 3|}$$

Next, set up a number line to help craft a piecewise definition for $|x - 3|$.

Thus, if $x < 3$, then $|x - 3| = -(x - 3)$, and

$$f(x) = \frac{(x - 3)(x + 1)}{-(x - 3)} = -(x + 1).$$

On the other hand, if $x \geq 3$, then $|x - 3| = x - 3$, and

$$f(x) = \frac{(x - 3)(x + 1)}{x - 3} = x + 1.$$

Thus, a piecewise definition for f is

$$f(x) = \begin{cases} -(x+1), & \text{if } x < 3, \\ x+1, & \text{if } x \geq 3. \end{cases}$$

We can now use the piecewise definition to evaluate the limits. First,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} -(x+1) = -4.$$

Secondly,

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x+1) = 4.$$

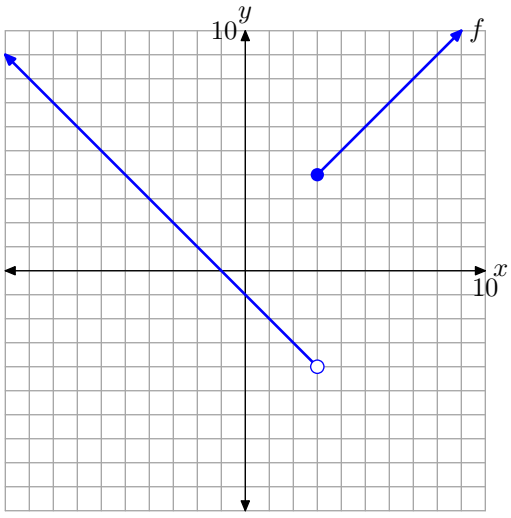
Exercise 1

(b) Without the aid of technology, use your piecewise definition to sketch the graph of f on graph paper. Label and scale each axis. You will be graded on accuracy, so be precise.

Solution: Using the piecewise definition

$$f(x) = \begin{cases} -(x+1), & \text{if } x < 3, \\ x+1, & \text{if } x \geq 3, \end{cases}$$

each piece of the graph will be a ray. Plotting the endpoint of the ray and a second point on the ray establishes the following graph.



Exercise 1

(c) Use the plot in part (b) to determine each of the limits in part (a). Use sound mathematical writing to justify your answer.

Solution: In the graph from the previous part, note that as x approaches 3 from the left, the y-values (function values) on the left-hand ray approach the number -4. Hence,

$$\lim_{x \to 3^-} f(x) = -4.$$

On the other hand, in the same graph, note that as x approaches 3 from the right, the y-values (function values) on the right-hand ray approach the number 4. Hence,

$$\lim_{x \to 3^+} f(x) = 4.$$

Exercise 1

EXERCISE 2. Suppose that the inequalities

$$\frac{1}{2} - \frac{x^2}{24} < \frac{1-\cos x}{x^2} < \frac{1}{2} \tag{2}$$

hold for values of x close to zero. What, if anything, does this tell you about the limit

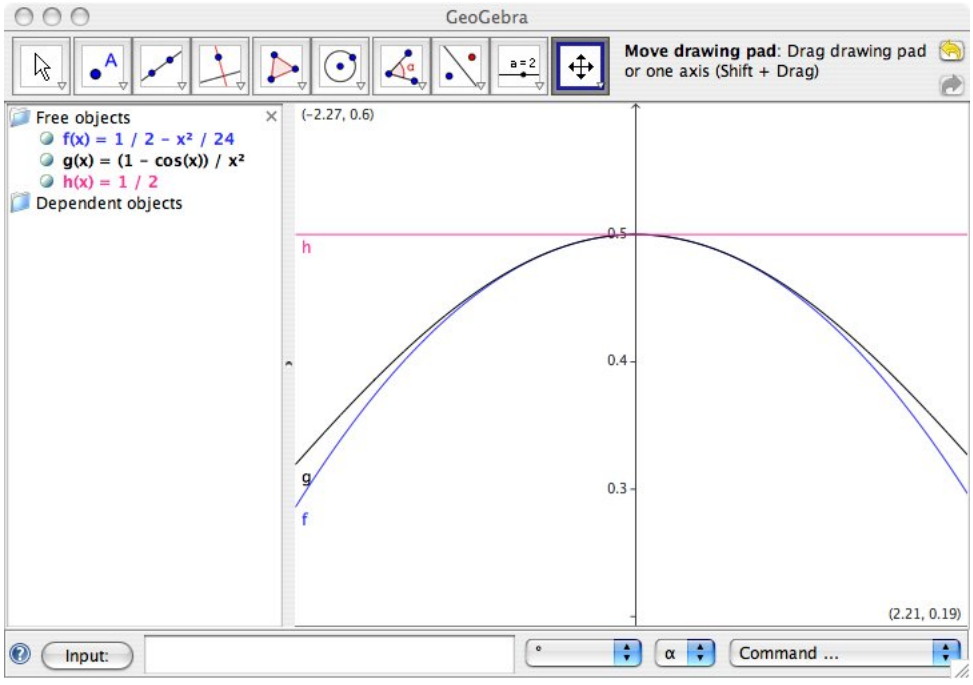
$$\lim_{x\to 0} \frac{1-\cos x}{x^2}. \tag{3}$$

Use Geogebra or Winplot to plot the inequalities and sound mathematical writing to explain and justify your answer. Be sure to quote any theorems you use to arrive at your solution.

Solution: In Geogebra, sketch the graphs of

$$f(x) = \frac{1}{2} - \frac{x^2}{24},$$
$$g(x) = \frac{1-\cos x}{x^2}, \text{and}$$
$$h(x) = \frac{1}{2}.$$

This provides the Geogebra result shown in the following image.



The Squeeze Theorem states that if

$$f(x) \le g(x) \le h(x),$$

and

$$\lim_{x\to a} f(x) = L \qquad \text{and} \qquad \lim_{x\to a} h(x) = L,$$

then

$$\lim_{x\to a} g(x) = L.$$

In the graph, note that both f and h approach $1/2$ as x approaches zero. Because the graph of g is trapped between the graphs of f and h,

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}.$$

You can also make this argument analytically. Note that

$$\lim_{x\to 0} \left(\frac{1}{2} - \frac{x^2}{24}\right) = \frac{1}{2},$$

and

$$\lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.$$

Because

$$\frac{1}{2} - \frac{x^2}{24} \leq \frac{1-\cos x}{x^2} \leq \frac{1}{2},$$

by the Squeeze Theorem we must also have

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}.$$

Exercise 2

Exercise 3. Consider the function

$$f(x) = x + \sqrt{x^2+5x}. \tag{4}$$

(a) Determine

$$\lim_{x\to -\infty} f(x),$$

using hand calculations only.

Solution: In $f(x) = x + \sqrt{x^2+5x}$, it's tempting to argue that the first term goes to minus infinity and the second goes to plus infinity as $x \to -\infty$, making the sum equal to $-\infty + \infty = 0$. However, the limit property that says "The limit of a sum is the sum of the limits" can only be applied if the limits of each term exist and are finite. So, we need another approach.

Our approach will be to rationalize the numerator.

$$\begin{aligned}
\lim_{x\to -\infty} f(x) &= \lim_{x\to -\infty} \left(x + \sqrt{x^2+5x}\right) \\
&= \lim_{x\to -\infty} \frac{(x+\sqrt{x^2+5x})(x-\sqrt{x^2+5x})}{x-\sqrt{x^2+5x}} \\
&= \lim_{x\to -\infty} \frac{x^2-(x^2+5x)}{x-\sqrt{x^2+5x}} \\
&= \lim_{x\to -\infty} \frac{-5x}{x-\sqrt{x^2+5x}}.
\end{aligned}$$

We now factor an x^2 from the radical in the denominator, recalling that $\sqrt{x^2} = |x|$.

$$\begin{aligned}
\lim_{x\to -\infty} \frac{-5x}{x-\sqrt{x^2+5x}} &= \lim_{x\to -\infty} \frac{-5x}{x-\sqrt{x^2}\sqrt{1+5/x}} \\
&= \lim_{x\to -\infty} \frac{-5x}{x-|x|\sqrt{1+5/x}}
\end{aligned}$$

Because we are taking the limit as x approaches $-\infty$, the values of x are negative and $|x| = -x$. Hence,

$$\begin{aligned}\lim_{x\to-\infty} \frac{-5x}{x - |x|\sqrt{1+5/x}} &= \lim_{x\to-\infty} \frac{-5x}{x - (-x)\sqrt{1+5/x}} \\ &= \lim_{x\to-\infty} \frac{-5x}{x + x\sqrt{1+5/x}}\end{aligned}$$

We can now divide both numerator and denominator of this last fraction by x, obtaining

$$\begin{aligned}\lim_{x\to-\infty} \frac{-5x}{x + x\sqrt{1+5/x}} &= \lim_{x\to-\infty} \frac{-5}{1 + \sqrt{1+5/x}} \\ &= \frac{-5}{1+\sqrt{1+0}} \\ &= -\frac{5}{2}.\end{aligned}$$

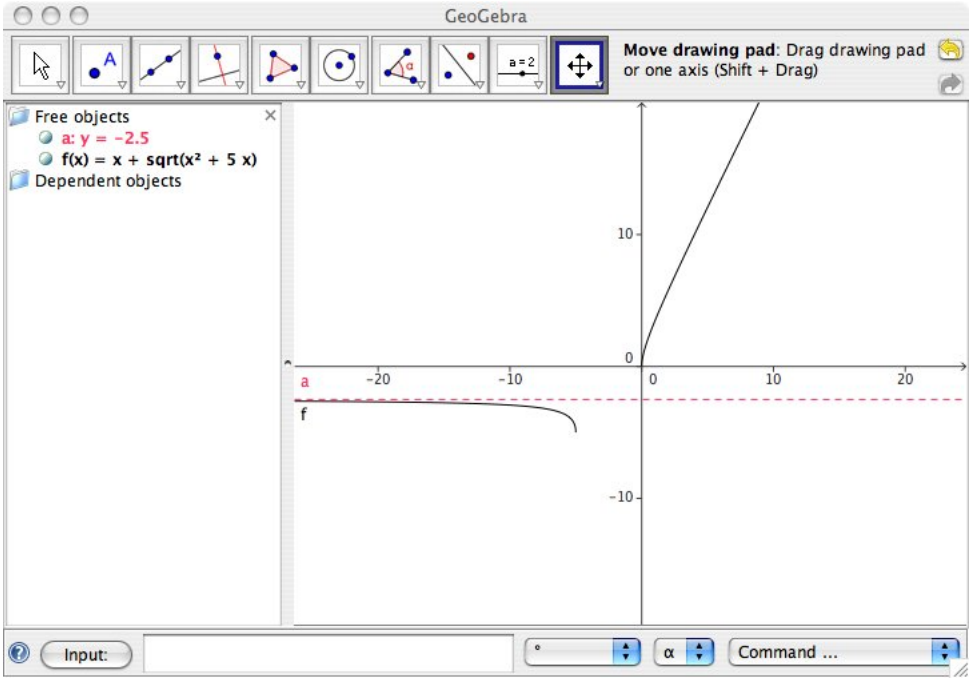
Exercise 3

(b) Use Geogebra or Winplot to sketch the graph of f and the horizontal asymptote determined in part (a).

Solution: Enter the equation

$$f(x) = x + \sqrt{x^2 + 5x}$$

into *Geogebra*. Add the horizontal asymptote $y = -5/2$ to obtain the following image.



Note in the image that as $x \to -\infty$, the graph on the left approaches the asymptote $y = -5/2$. This is evidence that

$$\lim_{x\to-\infty} (x + \sqrt{x^2 + 5x}) = -\frac{5}{2}.$$

Exercise 3