① $a_n = \left(1+\frac{2}{n}\right)^n$

$$\ln a_n = n \ln\left(1+\frac{2}{n}\right) = \frac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}}$$

Note that both numerator and denominator approach zero as $n \to \infty$. Thus, 0/0 qualifies for l'Hopital.

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \frac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{D_n \ln\left(1+\frac{2}{n}\right)}{D_n \frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{1+2/n}\cdot\frac{-2}{n^2}}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{2}{1+2/n}$$

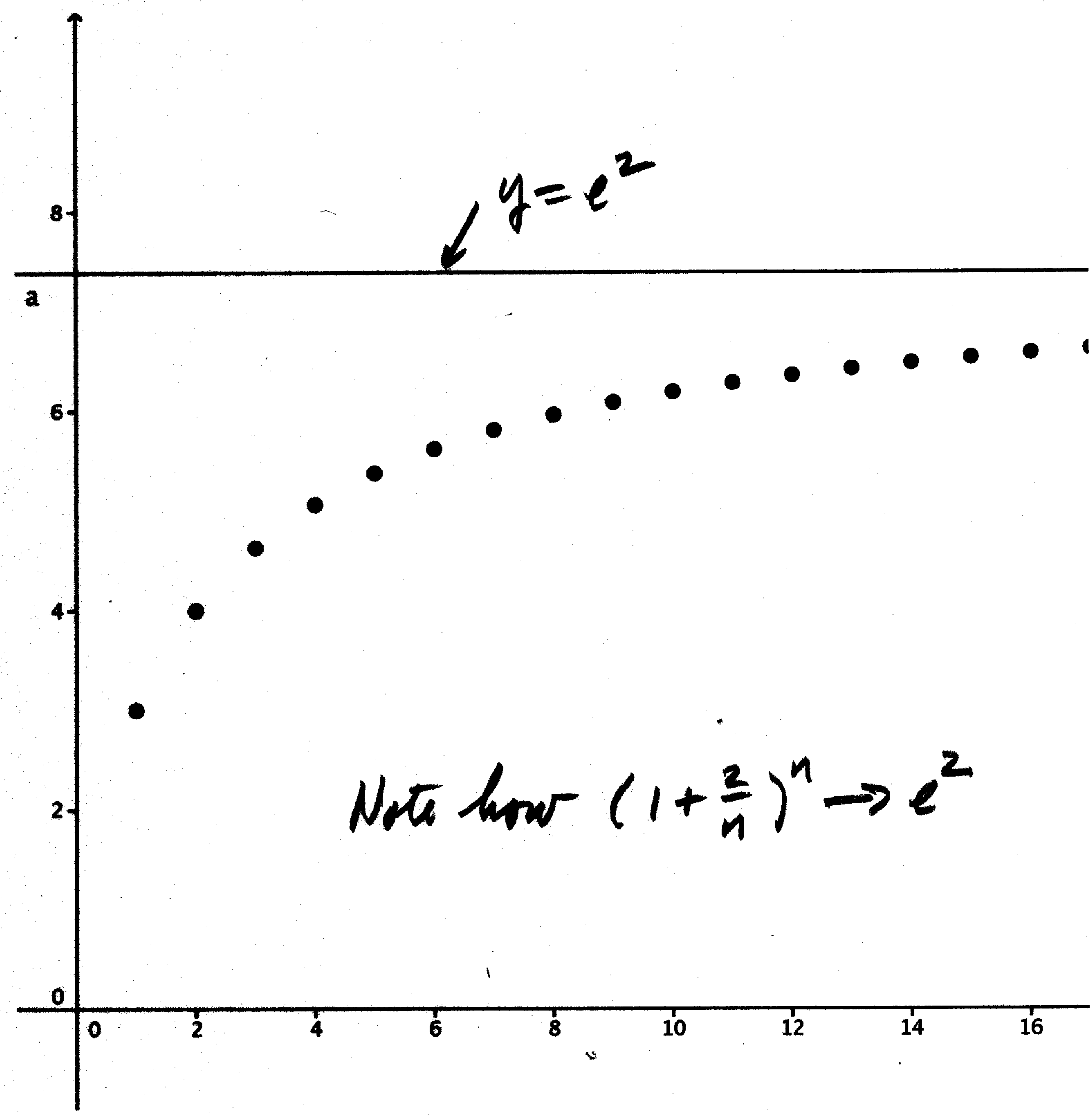
$$= 2$$

1 Therefore,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\ln a_n}$$

$$= e^{\lim_{n\to\infty} \ln a_n}$$

$$= e^{2}$$


$y = e^2$
a
8
6
4
2
0
0
2
4
6
8
10
12
14
16
Note how $(1+\frac{2}{n})^n \to e^2$

(2) $\sum_{n=0}^{\infty} 2\left(-\frac{1}{3}\right)^n = 2 - \frac{2}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$

The series is "geometric" with first term $a = 2$ and common ratio $r = -1/3$. Thus, the sum is

$$S = \frac{a}{1-r}$$

$$= \frac{2}{1-(-\frac{1}{3})}$$

$$= \frac{2}{\frac{4}{3}}$$

$$= 2 \cdot \frac{3}{4}$$

$$= 3/2$$

(3) Find a partial fraction decomposition:

$$\frac{1}{n^2+5n+6} = \frac{A}{n+3} + \frac{B}{n+2}$$

$$1 = A(n+2) + B(n+3)$$

$$n = -2 \Rightarrow 1 = B$$

$$n = -3 \Rightarrow 1 = -A \text{ or } A = -1$$

To find the infinite sum $\sum_{n=0}^{\infty} \frac{1}{n^2+5n+6}$, we must find the limit of the partial sums S_n. However,

$$S_n = \sum_{k=1}^{n} \frac{1}{k^2+5k+6} = \sum_{k=1}^{n} \left[\frac{1}{k+2} - \frac{1}{k+3}\right]$$

$$S_n = \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n+2} - \frac{1}{n+3}\right)$$

$$S_n = \frac{1}{3} - \frac{1}{n+3}$$

③ Therefore, if $S = \sum_{n=0}^{\infty} \frac{1}{n^2+5n+6}$, then

$$S = \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty} \left[\frac{1}{3} - \frac{1}{n+3}\right]$$

$$= \frac{1}{3}.$$

④ Consider

$$f(x) = \frac{1}{x(\ln x)^3}.$$

① Note that $f(x) > 0$ on $[2, \infty)$

② Note that f is continuous on $[2, \infty)$

③ Take a derivative:

$$\cancel{f(x) =} [x(\ln x)^3]^{-1}$$

$$f'(x) = -[x(\ln x)^3]^{-2}\{x \cdot 3(\ln x)^2 \cdot \frac{1}{x} + (\ln x)^3\}$$

$$f'(x) = -\frac{3(\ln x)^2 + (\ln x)^3}{[x(\ln x)^3]^2}$$

$$f'(x) = -\frac{(\ln x)^2(3 + \ln x)}{[x(\ln x)^3]^2}$$

$$f'(x) = \left[-\frac{(\ln x)^2}{[x(\ln x)^3]^2}\right](3 + \ln x)$$

Since the first factor is negative regardless of the value of x, $f'(x)$ will be negative when

$$3 + \ln x > 0$$

$$\ln x > -3$$

$$e^{\ln x} > e^{-3}$$

$$x > \frac{1}{e^3}.$$

So, $f'(x)$ is negative on $[2, \infty)$ so f is decreasing on $[2, \infty)$.

The hypotheses of the Integral Test are satisfied.

Now,

$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{1}{x(\ln x)^3}\,dx$$

Let $u = \ln x$, then $du = \frac{1}{x}\,dx$. Hence

$$\int \frac{1}{x(\ln x)^3}\,dx = \int \frac{1}{u^3}\,du$$

$$= \int u^{-3}\,du$$

$$= -\frac{1}{2}u^{-2}$$

$$= -\frac{1}{2u^2}$$

$$= -\frac{1}{2(\ln x)^2}$$

Hence,

$$\int_2^{\infty} \frac{1}{x(\ln x)^3}\,dx = \lim_{T\to\infty} \int_2^{T} \frac{1}{x(\ln x)^3}\,dx$$

$$= \lim_{T\to\infty} \left[-\frac{1}{2(\ln x)^2}\right]_2^T$$

$$= \lim_{T\to\infty} \left[-\frac{1}{2(\ln T)^2} + \frac{1}{2(\ln 2)^2} \right]$$

$$= \frac{1}{2(\ln 2)^2}.$$

Hence, $\int_2^\infty \frac{1}{x(\ln x)^3}\,dx$ converges. Therefore,

$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ converges.

(5) In $\sum_{n=1}^{\infty} \frac{3n^2+n}{n^4+\sqrt{n}}$, note that if we divide top and bottom by n^2, we get $\frac{3+\frac{1}{n}}{n^2+\frac{1}{n^{3/2}}}$, so it might make sense to compare this series to $\sum_{n=1}^{\infty} \frac{3}{n^2}$. So,

$$\lim_{n\to\infty} \frac{\frac{3n^2+n}{n^4+\sqrt{n}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{3n^2+n}{n^4+\sqrt{n}} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{3n^4+n^3}{n^4+\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{3+\frac{1}{n}}{1+\frac{1}{n^{7/2}}}$$

$$= 3 > 0$$

Since this limit is greater than zero, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series with $p=2$), then our series $\sum_{n=1}^{\infty} \frac{3n^2+n}{n^4+\sqrt{n}}$ also converges