Coupled Oscillators

Alex Gagen & Sean Larson

May 18, 2000

Abstract

We know that a single spring and mass system obeys simple harmonic motion (SHM). But how does a system of two spring and mass oscillators, coupled by a third spring behave? The analysis of this system of coupled oscillators is a great introduction to the concept of normal coordinates, and normal modes of vibration. For the normal modes of vibration, the motion is SHM. However, the general behavior of the two mass system is not SHM. It is a much more interesting motion in which the energy of the system oscillates back and forth between the two masses. The analysis here shows the general solution for the two mass system, in the undamped, undriven case.

Home Page
Title Page
Contents
◀◀ ▶▶
◀ ▶

Go Back
Full Screen
Close
Quit

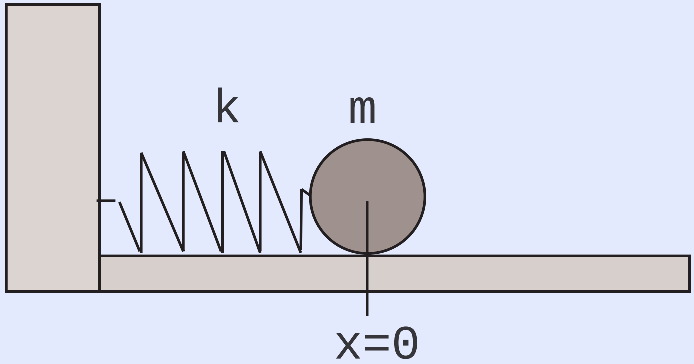

Figure 1: A Single Spring and Mass System

1. A Single Spring & Mass Oscillator

1.1. Description of the Model

Before we discuss the coupled oscillators, we will review the simple problem of a single spring and mass oscillator. This system consists of a single mass attached to a spring, and allowed to slide on a horizontal surface. A picture of this can be seen in Figure 1. For simplicity, we will ignore any damping forces acting on the mass, such as air resistance, or friction. With these assumptions, we need only consider the force that the spring exerts on the mass. We will assume that the spring obeys Hooke's law, and thus exerts a linear restoring force on the mass. This means that when the mass is displaced by a distance Δx from its

equilibrium position, the spring exerts a force $F_s = -k\Delta x$ on the mass. We will restrict our system to one-dimensional motion along the x-axis.

1.2. Finding the General Solution

We can describe the state of our system at any time in terms of the mass's displacement from equilibrium $x(t)$, and velocity $v(t) = \frac{dx}{dt}$. Given initial conditions for the system, $x(0)$ and $v(0)$, we would like to be able to solve for the masses position as a function of time.

Newton's Second Law

Newton's second law is an important tool in solving this problem as well as the more complex problem of the coupled oscillators. Newton's second law says that the sum of the forces acting on a body is equal to the product of the object's mass and its acceleration. Mathematically this is stated as, $F = ma$. In our case the displacement of the mass is $x(t)$, so the acceleration is $\frac{d^2x}{dt^2}$. So we state Newton's second law as:

$$F = m\frac{d^2x}{dt^2}$$

And we know that the net force acting on the mass is simply the force due to the spring, which we know to be:

$$F_s = -kx$$

So we write Newton's Second Law for our spring and mass system as:

$$F_s = -kx = m\frac{d^2x}{dt^2}$$

Which is rearranged:

$$mx'' + kx = 0$$

This equation is traditionally rewritten in the form:

$$x'' + \left(\frac{k}{m}\right) x = 0 \tag{1}$$

Now we make a simple substitution to write the equation in a simpler form.

$$\omega = \sqrt{\frac{k}{m}} \tag{2}$$

which is known as the natural frequency of the mass. The reason for this will be revealed later. Now we have the second order differential equation:

$$x'' + \omega^2 x = 0 \tag{3}$$

This is the final form of our differential equation describing the motion of the mass.

The Method of the Lucky Guess

We will solve this equation in a way that at first seems to be a bit of a swindle, but in fact is a common technique for solving differential equations. We guess a solution. Though this is not a completely random guess, but rather, an educated guess. Our guess is:

$$x = e^{\lambda t}$$

Home Page
Title Page
Contents
◀◀ ▶▶
◀ ▶
Page 5 of 21
Go Back
Full Screen
Close
Quit

where λ must be found. Taking the first and second derivatives of our guess, we get:

$$x' = \lambda e^{\lambda t}$$

and

$$x'' = \lambda^2 e^{\lambda t}$$

which we substitute back into Equation 2 and simplify, giving us the characteristic polynomial $p(\lambda)$.

$$p(\lambda) = \lambda^2 + \omega^2 = 0 \tag{4}$$

which we solve for λ

$$\lambda = \pm i\omega$$

and obtain our solution.

$$x = e^{i\omega t} \tag{5}$$

But this is not the form we want it in. So we expand it using Euler's formula.

$$x = \cos \omega t + i \sin \omega t \tag{6}$$

Now we use a theorem from differential equations that states that for complex valued solutions of a linear differential equation, both the real and imaginary parts of the complex solution, are solutions. And we form our general solution by taking a linear combination of these two solutions.

$$x(t) = C_1 \cos \omega t + C_2 \sin \omega t \tag{7}$$

Using trigonometry, we can rewrite our general solution in a more useful form:

$$x = A \cos(\omega t - \phi) \tag{8}$$

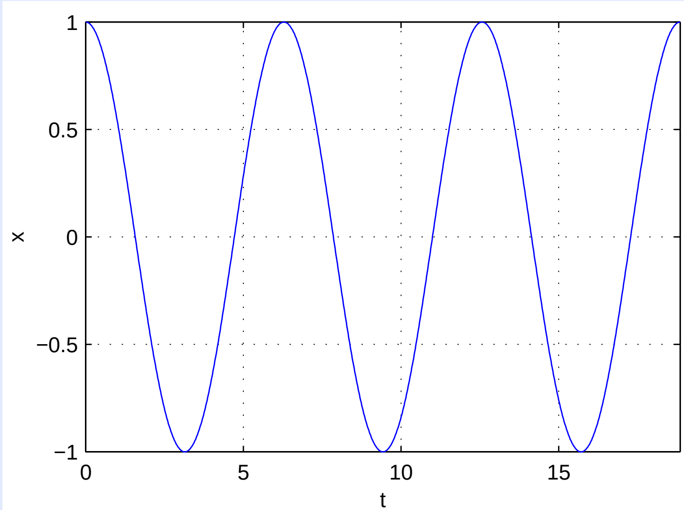

Figure 2: Simple Harmonic Motion: $x(t) = A\cos(\omega t - \phi), A = 1, \omega = 1, \phi = 0$

1.3. Analysis of Results

From our solution, we now know that the masses motion has an amplitude of A, an angular frequency of ω, and a phase shift of ϕ radians. Now we can see why the substitution for ω in Equation 2 was made. As we can see in Equation 8, the natural frequency ω is the angular frequency of the masses motion. A graph of the mass's motion where $A = 1$, $\omega = 1$, and $\phi = 0$ can be seen in Figure 2.

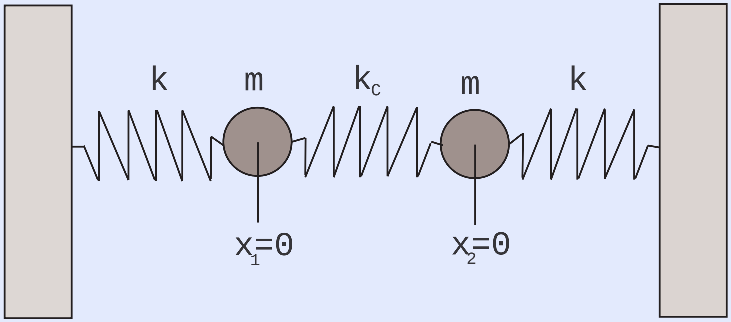

Figure 3: Coupled Oscillators

2. Coupled Oscillators

2.1. Description of the Model

Now that we have reviewed the concepts of simple harmonic motion, we are ready to discuss the problem of the coupled oscillators. As seen in Figure 3, we have two spring and mass oscillators with stiffness constants k, and masses m coupled together by a third spring of stiffness k_c. Also it should be noted that $k_c << k$. So we can say that our system is weakly coupled. As we did with the single spring and mass system, we will ignore any damping forces, such as friction or air-resistance. And the motion of the masses is again restricted to one dimension. We will make the same assumption that the springs obey Hooke's law and thus exert a linear restoring force on the masses given by $F_s = -k\Delta x$.

Home Page
Title Page
Contents
◀◀ ▶▶
◀ ▶
Page 8 of 21
Go Back
Full Screen
Close
Quit

2.2. Finding the General Solution

Now we would like to find the general solution for the motion of the two masses. Their positions are given by $x_1(t)$ and $x_2(t)$.

Newton's Second Law

As we did for the spring and mass system, we apply Newton's second law to the two mass system to obtain two second order differential equations describing the motion of the masses.

$$
\begin{aligned}
mx_1'' + kx_1 - k_c(x_2 - x_1) &= 0 \\
mx_2'' + kx_2 - k_c(x_1 - x_2) &= 0
\end{aligned}
$$

We rewrite these in the traditional form.

$$x_1'' + \left(\frac{k + k_c}{m}\right) x_1 - \left(\frac{k_c}{m}\right) x_2 = 0 \tag{9}$$

$$x_2'' + \left(\frac{k + k_c}{m}\right) x_2 - \left(\frac{k_c}{m}\right) x_1 = 0 \tag{10}$$

Note that these equations cannot be easily solved as we did for the single spring and mass. Each equation has a coupling term, which complicates matters, and prevents us from solving each equation separately. To make these equations easily solvable, we switch from our coordinates x_1 and x_2, to coordinates q_1, and q_2. These new coordinates are known as normal coordinates. To make this change, we first add the two equations together, and then subtract them, to

obtain two new equations. Adding Equation 9 and Equation 10, we obtain:

$$(x_1 + x_2)'' + \left(\frac{k_c}{m}\right)(x_1 + x_2) = 0 \tag{11}$$

Subtracting Equation 9 and Equation 10, we obtain:

$$(x_1 - x_2)'' + \left(\frac{k + 2k_c}{m}\right)(x_1 - x_2) = 0 \tag{12}$$

Now we change to our normal coordinates q_1 and q_2. We will also make a substitution for ω_1 and ω_2, the normal frequencies of the system. It turns out that these normal frequencies are the frequencies at which the masses oscillate in their normal modes of vibration. This will be further explained later.

$$q_1 = x_1 + x_2 \qquad \omega_1 = \sqrt{\frac{k}{m}} \tag{13}$$

$$q_2 = x_1 - x_2 \qquad \omega_2 = \sqrt{\frac{k + 2k_c}{m}} \tag{14}$$

Which we substitute into Equation 11 and Equation 12.

$$q_1'' + \omega_1^2 q_1 = 0 \tag{15}$$

$$q_2'' + \omega_2^2 q_2 = 0 \tag{16}$$

Now we have our equations in a much simpler form. In fact these equations are exactly the same as Equation 3, for the single spring and mass oscillator. Since

Home Page
Title Page
Contents
Page 10 of 21
Go Back
Full Screen
Close
Quit

they are not coupled, they can be solved separately in the same way Equation 3 was. The solution is therefore:

$$q_1(t) = C_1 \cos \omega_1 t + C_2 \sin \omega_1 t \tag{17}$$

$$q_2(t) = C_3 \cos \omega_2 t + C_4 \sin \omega_2 t \tag{18}$$

Now we use the relationship

$$x_1 = \frac{q_1 + q_2}{2}$$

$$x_2 = \frac{q_1 - q_2}{2}$$

to give the general solution for x_1 and x_2.

$$x_1(t) = C_1 \cos \omega_1 t + C_2 \sin \omega_1 t + C_3 \cos \omega_2 t + C_4 \sin \omega_2 t \tag{19}$$

$$x_2(t) = C_1 \cos \omega_1 t + C_2 \sin \omega_1 t - C_3 \cos \omega_2 t - C_4 \sin \omega_2 t \tag{20}$$

2.3. Behavior of the System

Now that we have the general solution for our coupled oscillators, we can give our system some initial conditions, and solve for the motion of the masses. Let us assume that we displace m_1 by a distance A, and m_2 by a distance B, and the masses are released from rest. This is the most general case for the masses motion. So our initial conditions are stated as:

$$x_1(0) = A \qquad x_1'(0) = 0$$

$$x_2(0) = B \qquad x_2'(0) = 0$$

After substituting these values into Equation 19 and Equation 20 we can solve for the constants C_1, C_2, C_3, and C_4.

$$C_2 = C_4 = 0$$
$$C_1 = \frac{A+B}{2}$$
$$C_3 = \frac{A-B}{2}$$

So our solution is:

$$x_1 = \frac{A+B}{2}\cos\omega_1 t + \frac{A-B}{2}\cos\omega_2 t \tag{21}$$
$$x_2 = \frac{A+B}{2}\cos\omega_1 t - \frac{A-B}{2}\cos\omega_2 t \tag{22}$$

While these equations give us a solution that we can graph to see the behavior of the masses, we would like to have our solution in a form that shows more clearly how the masses will behave. First we will consider the motion of the first mass, x_1. We substitute $e^{i\omega t}$ in for $\cos\omega t$.

$$x_c = \frac{1}{2}\left[(A+B)e^{i\omega_1 t} + (A-B)e^{i\omega_2 t}\right]$$

and we remember that $x_1 = Re(x_c)$. Now we will make a substitution for ω_1 and ω_2 and use the properties of exponents to further simplify our equation.

$$\alpha = \frac{\omega_1+\omega_2}{2} \qquad \beta = \frac{\omega_1-\omega_2}{2} \tag{23}$$
$$\Rightarrow \omega_1 = \alpha+\beta \qquad \omega_2 = \alpha-\beta \tag{24}$$

Which gives us:

$$x_c = \frac{1}{2}\left[(A+B)e^{i(\alpha+\beta)t} + (A-B)e^{i(\alpha-\beta)t}\right]$$

$$\Rightarrow x_c = \frac{1}{2}e^{i\alpha t}\left[(A+B)e^{i\beta t} + (A-B)e^{-i\beta t}\right]$$

Using Euler's formula, we expand the equation:

$$x_c = \frac{1}{2}(\cos\alpha t + i\sin\alpha t)((A+B)(\cos\beta t + i\sin\beta t) + (A-B)(\cos\beta t - i\sin\beta t))$$

This is simplified to:

$$x_c = A\cos\alpha t\cos\beta t - B\sin\alpha t\sin\beta t + i(A\cos\beta t\sin\alpha t + B\cos\alpha t\sin\beta t)$$

Now we use the fact that $x_1 = Re(x_c)$.

$$x_1 = A\cos\alpha t\cos\beta t - B\sin\alpha t\sin\beta t \tag{25}$$

In a similar manner, x_2 is found to be:

$$x_2 = B\cos\alpha t\cos\beta t - A\sin\alpha t\sin\beta t \tag{26}$$

With these solutions it is easier to interpret the motion of the masses.

2.4. Normal modes

There are two normal modes of vibration for this system. The symmetric normal mode occurs when both masses are displaced an equal amount in the same direction and released from rest. The non-symmetric mode occurs when both masses are displaced an equal distance from their equilibrium positions but in opposite directions.

Home Page
Title Page
Contents
Page 13 of 21
Go Back
Full Screen
Close
Quit

The Symmetric Mode

First we will consider the symmetric mode of vibration where $A = B$. So our initial conditions are:

$$x_1(0) = A \qquad x_1'(0) = 0$$
$$x_2(0) = A \qquad x_2'(0) = 0$$

So Equation 25 and Equation 26 become:

$$x_1 = A(\cos \alpha t \cos \beta t - \sin \alpha t \sin \beta t)$$
$$x_2 = A(\cos \alpha t \cos \beta t - \sin \alpha t \sin \beta t)$$

Notice that $x_1 = x_2$. We then use the Addition Formula from trigonometry:

$$x_1 = x_2 = A \cos(\alpha t + \beta t)$$

This is further simplified by replacing α (Equation 23) and β (Equation 24) with ω_1.

$$x_1 = x_2 = A \cos \omega_1 t \tag{27}$$

This is simple harmonic motion of frequency ω_1. This can be understood by realizing that the coupling spring is never stretched or compressed. In effect we have two independent spring and mass systems oscillating in SHM with their natural frequencies. A graph of the motions of the two masses can be seen in Figure 4. Notice that both masses have exactly the same motion.

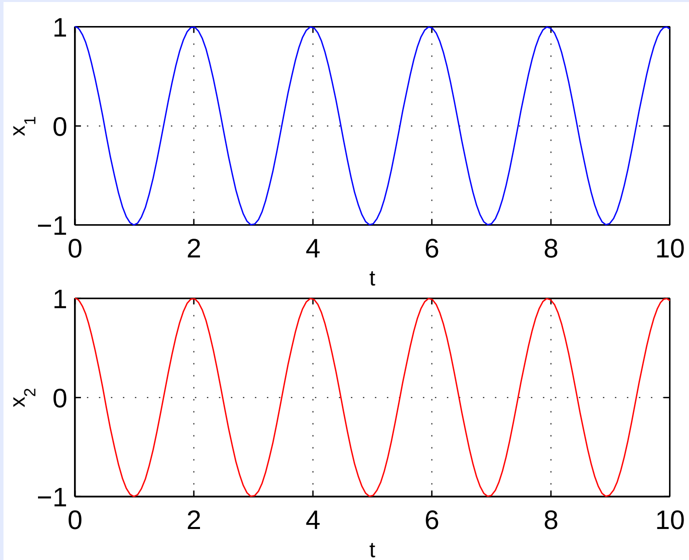

Figure 4: The Symmetric Mode: $x_1 = x_2 = A \cos \omega_1 t$

The Anti-Symmetric Mode

Now we consider the second normal mode, known as the anti-symmetric mode. We displace each mass by an equal distance in opposite directions. Our initial conditions are:

$$x_1(0) = A \qquad x_1'(0) = 0$$
$$x_2(0) = -A \qquad x_2'(0) = 0$$

which gives us the equations of motion

$$x_1(t) = A(\cos\alpha t \cos\beta t - \sin\alpha t \sin\beta t)$$
$$x_2(t) = -A(\cos\alpha t \cos\beta t - \sin\alpha t \sin\beta t)$$

Notice

$$x_1 = -x_2$$

The motion of the masses is symmetric but opposite. We replace α and β with ω_2 and arrive at the solution:

$$x_1 = A\cos\omega_2 \qquad (28)$$
$$x_2 = -A\cos\omega_2 \qquad (29)$$

In the anti-symmetric mode, the masses oscillate with a frequency ω_2, while the masses oscillated with a frequency ω_1 for the symmetric mode of vibration. $\omega_2 > \omega_1$ (Equation 13 and Equation 14), which means that for the second normal mode, the masses are oscillating more rapidly. This is caused by the coupling spring, which was not affecting the masses for the symmetric mode. A graph of the motion can be seen in Figure 5. Notice that the two masses have exactly opposite motions. Their motion is again SHM of frequency ω_2.

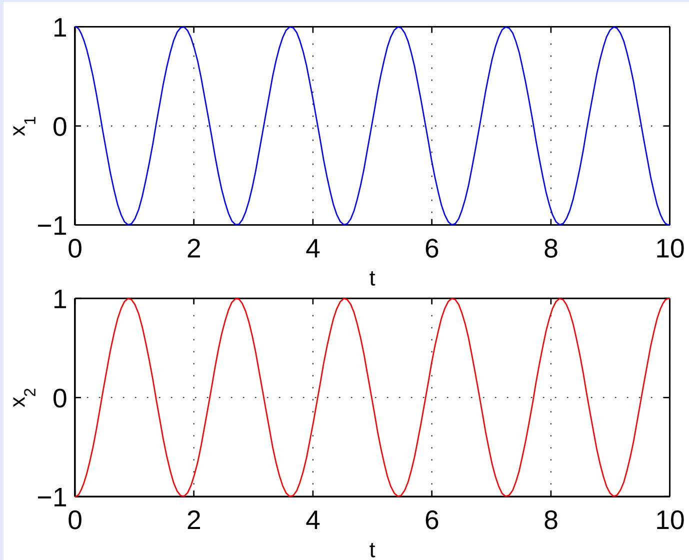

Figure 5: The Anti-Symmetric Mode: $x_1 = -x_2 = A\cos\omega_2$

General Behavior

Now that we have completed our analysis of the two normal modes of the system, we can discuss the more general behavior of the system where no symmetry is present. Let us give our system the initial conditions:

$$x_1(0) = A \qquad x_1'(0) = 0$$
$$x_2(0) = 0 \qquad x_2'(0) = 0$$

By substituting these values into our general solution (Equation 25 and Equation 26) we get:

$$x_1 = A \cos \alpha t \cos \beta t \tag{30}$$
$$x_2 = -A \sin \alpha t \sin \beta t \tag{31}$$

We can now clearly see that m_1 and m_2 display a beating phenomenon, where β is the beat frequency and α is the frequency of the rapid oscillations, and the amplitude is A. This can be seen in Figure 6. Notice that the peaks of x_1 correspond to the troughs of x_2. Each mass oscillates between the maximum amplitude A and an amplitude of zero. If we give the system initial conditions of $x_1(0) = A$ and $x_2(0) = B$ and the two masses are released from rest, we can see a similar behavior. This is shown in Figure 7. Again, the peaks of x_1 occur at the same times as the troughs of x_2. However, neither mass's oscillations ever reach an amplitude of zero. This is because neither mass started at its equilibrium position. Here each mass oscillates between the maximum amplitude A and the minimum amplitude B. The energy of the system is continuously oscillating between m_1 and m_2, while the total energy of the system remains constant.

A Single Spring & . . .
Coupled Oscillators
Conclusion
Home Page
Title Page
Contents
Page 18 of 21
Go Back
Full Screen
Close
Quit

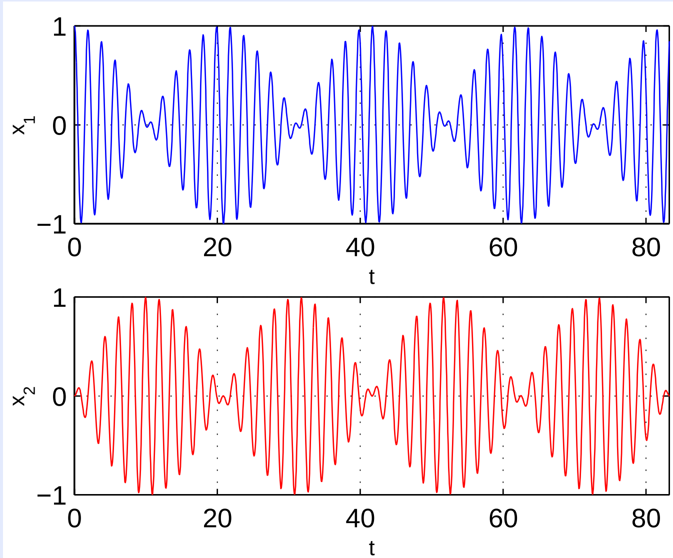

Figure 6: $x_1(0) = A, x_2(0) = 0$

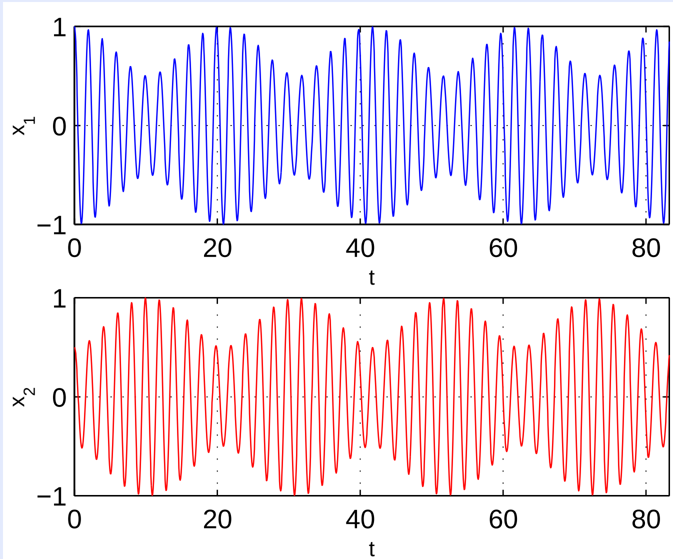

Figure 7: $x_1(0) = A, x_2(0) = B$

3. Conclusion

Having completed our analysis, we have found the normal modes of the system, as well as its normal frequencies. And we have determined the general behavior of the system. We can see that given two different initial displacements for m_1 and m_2, the energy of the system oscillates back and forth between the two masses continuously, never reaching a steady state of motion for either mass.

References

[A.P. French, 1971] French, *Vibrations and Waves*

[Blanchard, Devaney, Hall, 1998] Blanchard, Devaney, Hall, *Differential Equations*

[Dave Arnold, 1999] Arnold, *Ordinary Differential Equations using Matlab*