Modeling the Angular Velocity of a Spinning Tennis Racket

Lynn V. McIndoo
Jody S. Hourigan

May 20, 1999

Abstract

Abstract. To model the rotational stability of a spinning tennis racket about its 3 perpendicular axes.

1. Introduction

We introduce our topic by first observing the rotational behavior of a spinning tennis racket about each of its perpendicular axes. Along two of the axes, the racket spins steadily but on the third axis the racket wobbles considerably. We are interested in finding out why this is so. In order to do this, we model this conservative system and then closely examine the stability properties about each of the equilibrium points. In doing so, we gain a greater understanding of the behavior exhibited in this system.

2. Classification of Varying Equilibrium Points

We will begin our discussion by introducing the different possibilities of equilibrium points found in autonomous systems of ordinary differential equations. Let us begin with asymptotically stable equilibrium points. To help understand the behavior around this type of equilibrium point, imagine a pendulum bob (the mass) hanging down at rest. If one were to lift the bob from its initial position and then release it, the bob would oscillate about its initial position and eventually its velocity would decay to zero. In other words, the bob will return to its original rest position until perturbed again.

Let us now consider another scenario. Once again imagine a pendulum bob. This time however, the pendulum is balanced at rest directly above, 180° from the previous equilibrium position. With the very slightest perturbation, the bob would drop from its rest position, never returning. This particular case describes an unstable equilibrium point.

Finally, try to imagine a pendulum which experiences no forces of friction whatsoever. Release the bob, at say, and angle of 45° and watch how the bob oscillates without decay about its downward equilibrium position forever. This example is described as neutrally stable.

Now that we have informally described these three possible equilibrium point behaviors, let us now turn towards defining them in mathematical terms.

2.1. Defining Stability

The system $x' = f(x)$ is defined to be stable at an equilibrium point p if for each positive number ε there is a positive number δ such that: If

$$\|x^o - p\| < \varepsilon, \quad \text{for all } \; t \geq 0$$

The system is asymptotically stable at p if it is stable at p, and

$$\|x(t, x^o) - p\| \longrightarrow 0$$

Or in other words,

$$x(t, x^o) \longrightarrow p \text{ as } t \longrightarrow +\infty$$

for all points x^onear p. The system is neutrally stable at p, but not asymptotically stable. Finally, the system is unstable at p if it is not stable at p. So stability means that if an orbit starts within distance δ of the equilibrium point p, from then on the orbit stays within distance $\varepsilon$of p.

An equilibrium point is described as an attractor if

$$\|x(t, x^o) - p\| \longrightarrow 0 \;\; \text{as } t \longrightarrow +\infty$$

for all x^o near p. Therefore, let us further clarify our these terms. A system is described as asymptotically stable at p if it is stable at p and an attractor. The system is considered neutrally stable if it is stable at p and not an attractor. To be unstable, means that for some number ε_o, there are points q that are arbitrarily close to p with the property that at some point at a later time T the point $x(T, q)$ is farther away from p than the distance ε_o.

Now let us define the difference between global and a local asymptotically stable equilibrium point. The basin of attraction of an asymptotically stable equilibrium point p is the set of all points x^o such that

$$\|x(t, x^o) - p\| \longrightarrow 0 \text{ as } t \longrightarrow +\infty$$

However, the asymptotic stability at p is global is the basin of attraction includes all of space, and is local otherwise. Therefore, a system may possess many different equilibrium points as we shall see.

2.2. Stability Properties of a Linear System

Suppose that A is an $n \times n$ matrix of real constants. For every eigenvalue $\lambda$of A, suppose that m_λ denotes a multiplicity of λ and d_λ the dimension of $\lambda's$ eigenspace. Then:

1. The system $x' = Ax$ is globally asymptotically stable at the origin if and only if every eigenvalue of A has a negative real part.
2. The system $x' = Ax$ is neutrally stable at the origin if and only if
   - Every eigenvalue of A has a nonpositive real part, and
   - At least one eigenvalue has a zero part, and $d_\lambda = m_\lambda$ for every eigenvalue λ with a zero real part.
3. The system $x' = Ax$ is unstable at the origin if and only if
   - Some eigenvalue of A has a positive real part, or
   - There is an eigenvalue λ with a zero part and $d_\lambda < m$.

2.3. Stability of Nearly Linear Systems

What is a nearly linear system? Suppose that A is an $n \times n$ matrix of real constants. Suppose also that $P(x)$ is a function vector that is continuously differentiable in an open ball $B_r(p)$ in R^n, that $P(p) = 0$, and that $P(x)$ is at least second order at P. Then the nearly linear system

$$x' = A(x - p) + P(x) \tag{2.1}$$

has the following stability properties:

1. The system is asymptotically stable at p if all eigenvalues of A have negative real parts.

2. The system is unstable at p if A has an eigenvalue with a positive real part.

Therefore, one must apply a test in order to determine the stability properties of the nearly linear system of equation 2.1. The first step is the determine the signs of the real parts of the eigenvalues of A. With this information, conclude that the system either asymptotically stable or unstable. If all of the signs are negative, the system is asymptotically stable at p and if at least one of the signs is positive, the system is unstable at p. However, more information is needed if some eigenvalue has a zero real part but none has a positive real part.

3. The Jacobian Matrix

How do we find the stability at point p if we do not have a linear or a nearly linear system? This case is called a nonlinear system and the way we handle these are by the use of the Jacobian matrix. Given the system

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned}$$

The Jacobian matrix linearizes the system near the equilibrium point (x_o, y_o), and the resulting matrix equation may be expressed by

$$\begin{bmatrix} x \\ y \end{bmatrix}' = J(x_o, y_o) \begin{bmatrix} x \\ y \end{bmatrix}$$

Which is equivalent to

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}' = \begin{bmatrix} \frac{\partial f}{\partial x}(x_o, y_o) & \frac{\partial f}{\partial y}(x_o, y_o) \\ \frac{\partial g}{\partial x}(x_o, y_o) & \frac{\partial g}{\partial y}(x_o, y_o) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

4. When a System is Conserved

Consider again a pendulum bob, this time consider an undamped bob rotating about its pivot at a constant velocity. If one were to release the bob from a different position or initial velocity, this bob would orbit the pivot at a new velocity. In other words, the energy in the system described is being conserved. To better understand a conservative system we consider the planar autonomous systems possessing the form

$$\begin{aligned} x' &= g(y) \\ y' &= f(x) \end{aligned} \tag{4.1}$$

Let us define that

$$\begin{aligned} G(y) &= \text{ antiderivative of } g(y) \text{ and that} \\ F(x) &= \text{antiderivative of } f(x) \end{aligned}$$

Also, let us define the function K as

$$K(x, y) = G(y) - F(x)$$

This function is conserved along each solution

$$\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned}$$

For a system to be conserved the following must be true:

$$K(x(t), y(t))' = 0$$

We will show that our system is conserved by taking the derivative of K

$$\frac{d}{dt}K(x(t), y(t)) = \frac{\partial K}{\partial x}\frac{dx}{dt} + \frac{\partial K}{\partial y}\frac{dy}{dt}$$

According to equation 4.1 this is equivalent to:

$$-f(x)g(y) + f(x)g(y) = 0$$

Therefore K is conserved along the orbits

$$\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned}$$

Although the value of K stays constant for particular $x(t)$ and $y(t)$ solutions, K may take on a different constant value for different $x(t)$ and $y(t)$ orbits. Or in other words, the function K is nonconstant for the system considered in equation 4.1 since the value of K is dependent on the particular orbit. Also, each orbit lies on a level set (or contour curve) of K since

$$K(x, y) = C$$

for some constant C.

4.1. An Integral

$K(x)$ is an integral of the autonomous system

$$x' = f(x) \tag{4.2}$$

if the following are true:

- the derivative of $K(x)$ is zero for all orbits $x = x(t)$ corresponding to the system in equation 4.2.
- K is nonconstant on every ball in R^n.

4.2. Integrals and Orbits

Using the definition of an integral, we can say that the system in 4.2 is conserved if $K(x)$ is an integral of the system. As a result, each orbit of the system lies on the level set of K, every level set is a union of orbits, and any orbit that meets an integral surface must remain on that integral surface.

5. Conservative Systems and a Tennis Racket

If one were to spin a tennis racket about one of its three axes, L_1, L_2, L_3, one would note that to spin the tennis racket along one of the axes is nearly impossible. We will explain why this is so by taking another look at conservative systems. Let us designate the angular velocity ω along the three corresponding axes to be ω_1, ω_2, ω_3. When calculating the angular velocities, air resistance is ignored and consider only the principal inertias I_1, I_2, I_3, as the influencing parameters. A principal inertia measures the response of a body to attempts to get the body to spin about an axis and is a function of the body's mass and its shape. According to Newton's force laws we have the system

$$\begin{aligned} \omega_1' &= \frac{I_2 - I_3}{I_1}\omega_2\omega_3 \\ \omega_2' &= \frac{I_3 - I_1}{I_2}\omega_1\omega_3 \\ \omega_3' &= \frac{I_1 - I_2}{I_3}\omega_1\omega_2 \end{aligned} \tag{5.1}$$

The kinetic energy of angular rotation is given by

$$KE(\omega_1, \omega_2, \omega_3) = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$

Recall that from the definition of an integral that if

$$\begin{aligned} \frac{d}{dt}KE &= I_1\omega_1\omega_1' + I_2\omega_2\omega_2' + I_3\omega_3\omega_3' \\ &= (I_2 - I_3)\omega_1\omega_2\omega_3 + (I_3 - I_1)\omega_1\omega_2\omega_3 + (I_1 - I_2)\omega_1\omega_2\omega_3 = 0 \end{aligned}$$

then KE is an integral of the system, and the system is conservative. The ellipsoid integral surface in three space is called an inertial ellipsoid and is given by the equation

$$KE = C$$

for some positive valued constant C.

5.1. Conserved Systems and its Integral Surface

In order to graph the solutions to system 5.1 we must assign values to the parameters. Setting

$$I_1 = 2,\ I_2 = 1,\ I_3 = 3$$

our system becomes

$$\begin{aligned} \omega_1' &= -\omega_2\omega_3 \\ \omega_2' &= \omega_1\omega_3 \\ \omega_3' &= \frac{1}{3}\omega_1\omega_2 \end{aligned} \tag{5.2}$$

The equation of the integral becomes

$$KE(\omega_1, \omega_2, \omega_3) = \frac{1}{2}(2\omega_1^2 + \omega_2^2 + 3\omega_3^2)$$

To solve for the coordinates of the equilibrium points along each of the three axes of rotation we first set the integral function

$$KE = C$$

This corresponds to the system's kinetic energy being conserved at some positive valued constant C. For our analysis, we have chosen

$$C = 12$$

Setting the integral equal to this constant leads to:

$$\begin{aligned}
\frac{1}{2}(2\omega_1^2 + \omega_2^2 + 3\omega_3^2) &= 12 \\
2\omega_1^2 + \omega_2^2 + 3\omega_3^2 &= 24 \\
\frac{\omega_1^2}{12} + \frac{\omega_2^2}{24} + \frac{\omega_3^2}{8} &= 1 \\
\frac{\omega_1^2}{\left(2\sqrt{3}\right)^2} + \frac{\omega_2^2}{\left(2\sqrt{6}\right)^2} + \frac{\omega_3^2}{\left(2\sqrt{2}\right)^2} &= 1
\end{aligned}$$

The last equation is in the form of an ellipsoid, where the value in the denominator is the distance from the origin corresponding to the three axes. This is similar to finding the ellipse's major and minor axes in two space. Therefore, the coordinates of the equilibrium points along with their corresponding axes of rotation are given by:

$$\begin{array}{ll}
(\pm 2\sqrt{3}, 0, 0) & \text{corresponding to the } L_1 \text{ rotational axes} \\
(0, \pm 2\sqrt{6}, 0) & \text{corresponding to the } L_2 \text{ rotational axes} \\
(0, 0, \pm 2\sqrt{2}) & \text{corresponding to the } L_3 \text{ rotational axes}
\end{array} \tag{5.3}$$

The ellipsoid in Figure 5.1 is a representation of the conserved energy of the system in three space. Note that the orbits correspond to the equilibrium points found in (5.3).

But how can we test the stability properties of these equilibrium points? In attempts to determine the stability of each equilibrium point we linearize the

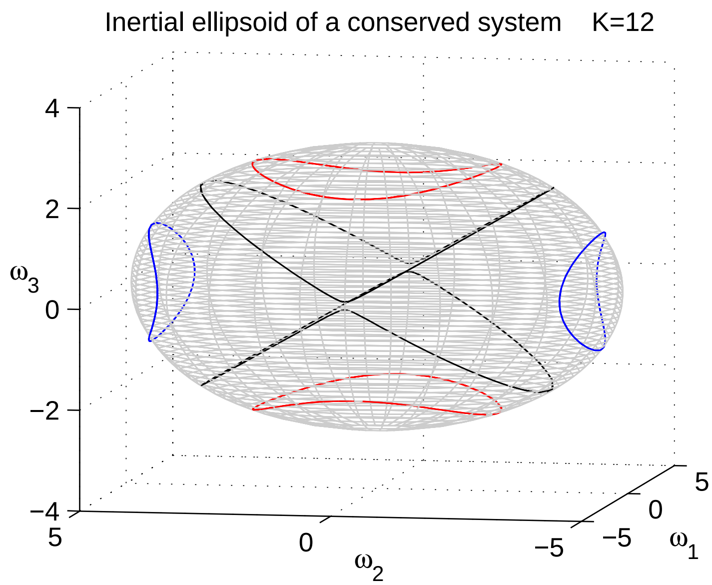

Figure 5.1: Integral Surface of the Conserved System

system by utilizing the Jacobian matrix. The Jacobian matrix of this system is represented by

$$\begin{bmatrix} 0 & -\omega_1 & -\omega_2 \\ \omega_3 & 0 & \omega_1 \\ \frac{1}{3}\omega_2 & \frac{1}{3}\omega_1 & 0 \end{bmatrix}_{(\omega_1,\omega_2,\omega_3)}$$

The Jacobian matrix evaluated at each of the three axes' equilibrium points yields matrices that posses the following eigenvalues

$$\lambda_1 = 2, \lambda_2 = -2, \text{ and } \lambda_3 = 0 \tag{5.4}$$

$$\lambda_1 = \frac{3363}{1189}i, \lambda_2 = -\frac{3363}{1189}i, \text{ and } \lambda_3 = 0 \tag{5.5}$$

$$\lambda_1 = \frac{3363}{1189}i, \lambda_2 = -\frac{3363}{1189}i, \text{ and } \lambda_3 = 0 \tag{5.6}$$

Where equation 5.4 corresponds to the L_1 axis, and equations 5.5 and 5.6 correspond to the L_2 and L_3 axes respectively. Equation 5.4 predicts a saddle behavior around its equilibrium point, while equations 5.5 and 5.6 predict a center around their corresponding equilibrium points. However, this analysis of the equilibrium points are not enough to absolutely determine the stability of these points for two reasons. First, due to the location of the center in the trace determinant plane, one must be cautious when a linearized system predicts a center. (This is because a center is positioned along the determinant axis, and to perturb this solution to the right or to the left may throw the solution into a sink or into a source. Because a linearization is in essence a "perturbation" of the solutions, it would be unwise to trust that what is actually occurring at an equilibrium point is in fact a center even though "center" behavior is predicted.) Another reason that we must turn to a different method for determining the stability of the equilibrium points is that each linearized system yielded an eigenvalue with zero value.

6. Lyapunov's Test for Stability

Lyapunov's test for stability is a very useful method of testing the stability properties of equilibrium points when other methods are inconclusive. Before one can utilize Lyapunov's test, we must first have an understanding of some definitions. Let us consider the function $V(x)$ (in the previous examples recall that this function is an integral of a system). $V(x)$ is positive definite if it is positively definite for all values except at the origin, and negatively definite is defined similarly. $V(x)$ is a strong Lyapunov function if V is positive definite while V' is negative definite. This leads us to Lyapunov's first test: suppose that there is a strong Lyapunov function $V(x)$, then the system is asymptotically stable at the origin. Since an equilibrium point for any function may be translated to the origin, this can test for asymptotic stability for any equilibrium point. The implications of this test do not apply to our system however. This is saying that the function

$$V(x) = C$$

must go to zero as t increases because V' is negative. As t increases, the function $V(x)$ will approach zero. Recall that the orbits of a conserved system lie on its integral surface. This means that the orbit $x(t)$ must lie on even smaller and smaller integral surfaces until the orbit sinks into the center of the integral surface which is the origin. We have defined a conserved system to be a system that has an integral, and a function equaling a constant when evaluated at the solutions the system. A visual interpretation of a conserved system is that each solution orbit remains on its integral surface. In the case of a strong Lyapunov function, the orbit $x(t)$ does not stay on any one integral surface $V(x)$. Therefore, because our system is conserved, there will be no asymptotically stable equilibrium points.

Now that we have the definition of a strong Lyapunov function, we need to familiarize ourselves with some definitions that will allow us to understand the implications of a weak Lyapunov function. If for some real-valued function $V(x)$ for which $V(0) = 0$ the following is true: $V(x)$ is positive semidefinite if $V(x) \geq 0$ on an open ball centered on the origin, and negative semidefinite is defined in a similar manner. V is indefinite if at the arbitrary points q_1, q_2

$$\begin{aligned} V(q_1) &< 0 \\ V(q_2) &> 0 \end{aligned}$$

$V(x)$ is a weak Lyapunov function if V is positive definite, but $V'(x)$ is only negative semidefinite. In other words, $V'(x) \leq 0$ near the origin. Suppose that there is a weak Lyapunov function, then the system is stable at the origin. Since our system dictates that we can not have an asymptotically stable equilibrium point, $V'(x) = 0$, for all t. Therefore, we have a neutrally stable equilibrium point because we can not cut across level sets of V. On the other hand, if both $V(x)$ and its derivative are both positive definite on a ball centered at the origin, then we have an unstable equilibrium point.

6.1. Using Lyapunov's Test for Stability on Our System

Now that we have defined some terms necessary for Lyapunov's test for stability, we shall now apply these tests to in order to better analyze our spinning tennis racket model. Let us return to (5.2) for the system of the angular velocity.

$$\begin{aligned} \omega_1' &= -\omega_2\omega_3 \\ \omega_2' &= \omega_1\omega_3 \\ \omega_3' &= \frac{1}{3}\omega_1\omega_2 \end{aligned}$$

We find two integrals of the system to be

$$\begin{aligned} V_1 &= \omega_1^2 + \omega_2^2 \\ V_2 &= \omega_1^2 + 3\omega_2^2 \end{aligned}$$

To verify these is a simple task. One needs only to check that

$$\begin{aligned} V_1{}' &= 0 \\ V_2{}' &= 0 \end{aligned}$$

following the motion of the particle satisfying the equations in system 5.2. We assign the function V as a combination of $V_1$and V_2 for the L_1 and L_2 axes respectively as

$$\begin{aligned} V &= (V_1 - \alpha)^2 + V_2 \\ V &= V_1 + (V_2 - \alpha)^2 \end{aligned}$$

V is a weak Lyapunov function at the equilibrium points

$$\begin{gathered} (0, \pm\alpha, 0) \\ (0, 0, \pm\alpha) \end{gathered}$$

Using the coordinates found in (5.3), we then substitute the equilibrium points corresponding to the L_2 and L_3 axes. They are

$$\begin{gathered} (0, \pm 2\sqrt{6}, 0) \\ (0, 0, \pm 2\sqrt{2}) \end{gathered} \tag{6.1}$$

respectively. Because these are weak Lyapunov functions, we therefore know from Lyapunov's stability tests that these are stable equilibrium points. In contrast, the equilibrium points

$$(\pm\alpha, 0, 0)$$

correspond to the L_1axis and the equilibrium points found there from (5.3) gives us

$$(\pm 2\sqrt{3}, 0, 0) \tag{6.2}$$

These points are unstable because they are always positive definite in our integral function

$$V = \omega_2\omega_3$$

We may verify this by

$$(\omega_2\omega_3)' = \omega_2\omega_3' + \omega_3\omega_2'$$

Substituting in our system of equations from 5.2 we obtain

$$\begin{aligned} (\omega_2\omega_3)' &= \omega_2(\frac{1}{3}\omega_1\omega_2) + \omega_3\,(\omega_1\omega_3) \\ (\omega_2\omega_3)' &= \omega_1(\frac{1}{3}\omega_2^2 + \omega_3^2) \end{aligned}$$

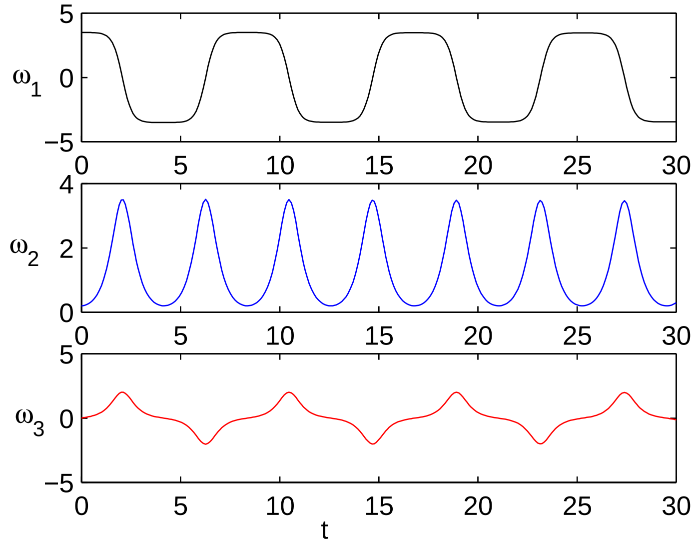

Figure 6.1: Angular Velocity vs. Time

which is positive definite on the ball at the equilibrium point found in (6.2). Therefore, we may conclude that this set of equilibrium points corresponding to the L_1axis are unstable. Now let us take a final look at the angular velocity of each axes of rotation versus time. The first graph corresponds to the unstable axis L_1. In Figure 6.1, note how the angular velocity ω_1 is nearly constant but then drops quickly to its negative counterpart. This periodic behavior corresponds to the wobbling observed on this axis. The remaining two graphs display a smoother periodic transition over time as we saw in their stable behavior about their respective axes.