The Optimal Height for Releasing Paratroopers

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Abstract

When paratroopers are released from a plane during combat they must obviously be dropped high enough so that there is enough time for the parachute to slow the descent and prevent injuries from the impact of landing. At the same time, however, they must be released low enough that they spend a minimal time in the air where they are helplessly exposed to withering enemy fire. The purpose of this paper is to determine the optimal height for releasing the paratroopers that will provide for a safe landing, while minimizing the time they are exposed to enemy ground fire.

1. Introduction

In order to understand the purpose of this paper it is first necessary to have a little background information on military parachuting. I will be discussing issues related to the mass aerial deployment of combat troops. During such deployments a continuous stream of soldiers are released simultaneously from each side of the aircraft, jumping out of the plane one after another like lemmings. Their parachutes are deployed by

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means of a 9.1 m long static line which connects their parachute to the plane and acts as a "rip-cord". In order to prevent injuries caused by impact, it is essential that the impact velocity of the paratrooper be between -4.6 and -5.2 m/s (About the equivalent of a jump from a $5'$ wall). The values are negative because I am taking the downward direction to be the negative direction. During their deployment and descent, a paratrooper is a high profile, relatively slow moving target for enemy ground forces. The paratroopers are effectively defenseless while in the air. It is therefore highly desirable to release the jumpers from a height that will allow sufficient time to decelerate, but minimal time in the air.

2. Modelling the Fall

In order to determine the proper height to release a paratrooper I set out to create a mathematical model that would correctly predict the behavior of a falling paratrooper. I started by outlining the variables to be utilized. Next I derived the mathematical equations to be used in the model. I then used the model to make some predictions. Lastly, I fine-tuned the model and made conclusions based on the collected data.

2.1. Description of Variables

The following is a list of variables that will be used in this paper to describe the motion of the falling paratrooper:

$$
\begin{aligned}
x(t) &= \text{Position at time } t \\
v(t) &= \text{Velocity at time } t \\
m &= \text{Mass of paratrooper, harness, and parachute} \\
g &= \text{Acceleration due to gravity} \\
k &= \text{Coefficient of drag} \\
t_d &= \text{Time of parachute deployment} \\
t_i &= \text{Time of paratrooper impact with the ground} \\
v_t &= \text{Terminal velocity with parachute deployed} \\
t_{d2} &= \text{Time of deployment of reserve parachute} \\
t_{i2} &= \text{Time of paratrooper impact using reserve parachute} \\
v_{t2} &= \text{Terminal velocity with reserve parachute deployed}
\end{aligned}
$$

2.2. Creating the Model

The first step in setting up the mathematical model for this problem is to establish the coordinate axis. Since I do not yet know the starting height, it makes sense to use my starting point, the door of the aircraft, as the origin. This will impact the calculations by making the position, velocity, and acceleration all negative.

The equations modelling the descent of a paratrooper can all be derived from Newton's second law of motion, which states that the sum of the forces acting on body are equal to the product of the mass of the body and its acceleration.

$$\sum F = ma$$

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In the case at hand, it can be written as

$$\text{Mass} \times \text{Acceleration} = \text{Force of Gravity} + \text{Air Resistance}$$

Acceleration is the derivative of velocity with respect to time. The force of gravity, being in the downward direction, is equal to $-(\text{mass} \times \text{acceleration due to gravity})$, or $-m \times g$. Air resistance is proportional to velocity and is represented by $k \times v$, where k is the coefficient of drag, and is opposite the direction of motion. This gives the equation

$$m\frac{dv}{dt} = -mg - kv$$

If we divide both sides by m, and remember that velocity is the derivative of position, we get the following system of equations

$$\begin{aligned} x'(t) &= v \\ v'(t) &= -g - \frac{k}{m}v \end{aligned} \tag{1}$$

At this point it is important to discuss k, the coefficient of drag. The coefficient of drag is determined by the shape of the falling body. In general, the more surface area a body has, the higher the coefficient of friction. Since the surface area of the paratrooper is significantly less than the surface area of the falling paratrooper once the parachute is deployed, we will denote k as the following piecewise function:

$$k = \begin{cases} k_1 & 0 \le t < t_d \\ k_2 & t \ge t_d \end{cases}$$

The Paratroopers will have the coefficient of drag, k_1, until the moment, t_d, that they reach the end of the 9.81m static line and the parachute is deployed. It is essential to determine the value of t_d so that k_2 can be substituted into the equation at the

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correct time. Before we can attempt to find the time that the paratrooper reaches the end of the static line, we must find the equation for position. We start with the equation for $v'(t)$ from Equation (1)

$$v'(t) = -g - \frac{k}{m}v$$

After factoring out k/m from the right side of the equation we have

$$\frac{dv}{dt} = -\frac{k}{m}\left(\frac{m}{k}g + v\right)$$

This is a separable differential equation that is easily solved. If you multiply both sides by dt, and divide both sides by the term $v + mg/k$, you are left with

$$\frac{dv}{v + \frac{m}{k}g} = -\frac{k}{m}dt$$

Next, we integrate both sides

$$\int \frac{1}{v + \frac{m}{k}g}dv = -\int \frac{k}{m}dt$$

$$\ln\left(v + \frac{m}{k}g\right) = -\frac{k}{m}t + C$$

Where C is an arbitrary constant of integration. We next take the exponential of both sides of the equation

$$v + \frac{m}{k}g = e^{-kt/m+C}$$

$$v = e^{C}e^{-kt/m} - \frac{m}{k}g$$

$$v = Ae^{-kt/m} - \frac{m}{k}g$$

Where the arbitrary constant A is substituted for e^C. The actual value of A is obtained by substituting the initial conditions into the equation. Since the starting position and the starting velocity are both 0, when we plug them into the equation for v from the last step we get

$$A = \frac{m}{k}g$$

If we plug this value for A back in, we get our equation for v

$$v(t) = \frac{m}{k}g\left(e^{-kt/m} - 1\right) \tag{2}$$

By integrating both sides, using a numerical solver, we get the equation for position

$$x(t) = \frac{mg}{k}\left(\frac{-m}{k}e^{-kt/m} - t\right) \tag{3}$$

Armed with these equations, we are now ready to calculate the optimal release height for the paratroopers.

2.3. Determining the Height of Release

By plugging 9.81 m (The length of the static line) into the equation (3), we obtain t_d. It is easy to see that, because of the factor of t in the exponential, solving for t by

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hand is virtually impossible, but by using a numerical solver it is soon determined that $t_d = 1.3626\,\mathrm{s}$. A more detailed explanation of how to use Matlab to do this calculation is provided in Appendix A.

Now that t_d is established we are able to calculate the velocity over the entire time span. At this point our object is to determine the time, $t_i \geq t_d$, at which the velocity is equal to $-5.2\,\mathrm{m/s}$. As previously stated, this is the maximum safe impact velocity for the paratroopers. Once we determine t_i, it is a simple matter to plug it in to Equation (3) to obtain the position at t_i . The absolute value of this position value is the optimal height at which to release the paratroopers.

In order to better visualize the motion of the paratrooper, our next step is to run the numerical solver using our initial conditions, $v(0) = 0$, switching from k_1 to k_2 at time t_d, and graph the results (See Figure 1). This graph shows that the paratrooper passes $-5.2\,\mathrm{m/s}$, on the way to terminal velocity. Using a numerical solver, t_i is determined to be approximately $3.7417\,\mathrm{s}$. If we plug this time value into Equation (3), and take the absolute value, we get a release height of approximately $24.8\,\mathrm{m}$. A quick look at Figure 2 confirms this. This is the absolute minimum height from which a paratrooper could be dropped and still land safely.

2.4. Analyzing the Answer

To non-paratroopers, $24.8\,\mathrm{m}$ seems like a frighteningly height from which to jump out of an aircraft. It is to paratroopers as well, and it would likely kill you. I have jumped from about $275\,\mathrm{m}$, and it did not seem like much more than $24.8\,\mathrm{m}$. The military would never release paratroopers from such a low height for two very important reasons. First, my model does not take into account any motion in the horizontal direction. Although a release height $24.8\,\mathrm{m}$ would give you plenty of time to decelerate in the vertical direction, you would still impact the ground with enough horizontal motion to cause severe injuries, if not death. Secondly, the military has wisely determined that

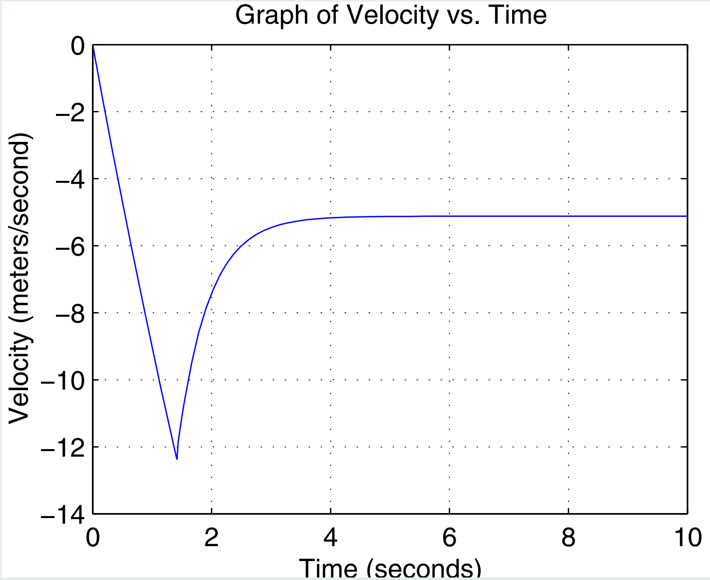

Figure 1: The graph shows the velocity passing through $-5.2\,\mathrm{m/s}$ on the way to a terminal velocity of approximately $5.12\,\mathrm{m/s}$

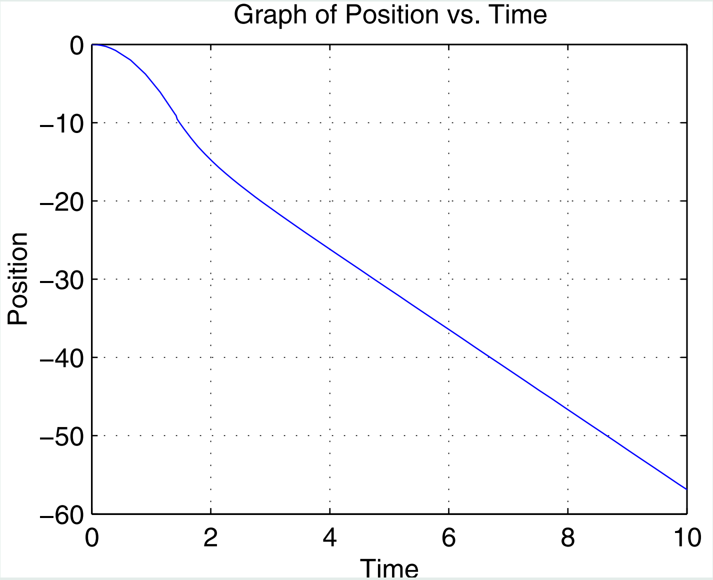

Figure 2: The graph shows the position at t_i is approximately $-24.8\,\mathrm{m}$

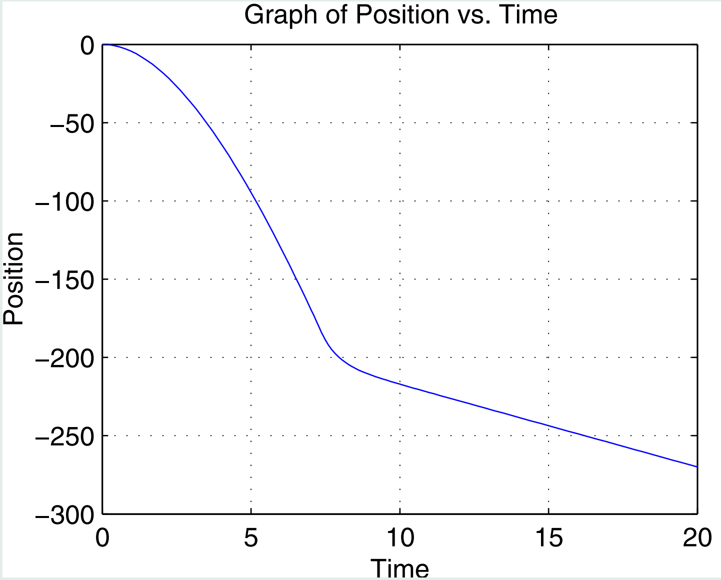

Figure 3: The graph shows the position at t_{i2} is approximately $-224.46\,\mathrm{m}$

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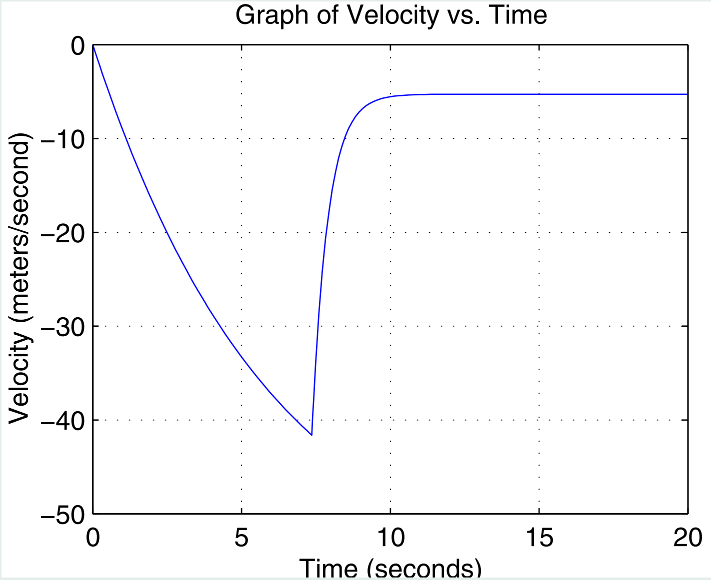

Figure 4: The graph shows the velocity passing through $-5.3\,\mathrm{m/s}$ on the way to a terminal velocity of approximately $-5.28\,\mathrm{m/s}$

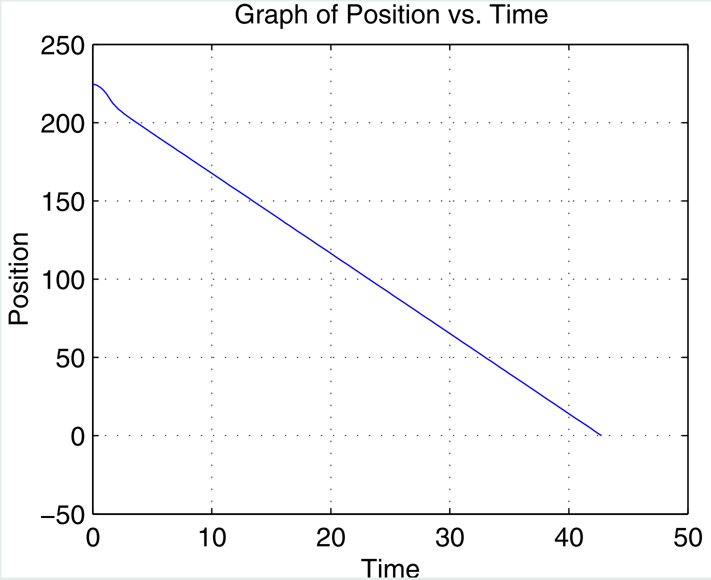

Figure 5: The graph shows the position is equal to 0 at approximately $t = 42.7525\,\mathrm{s}$

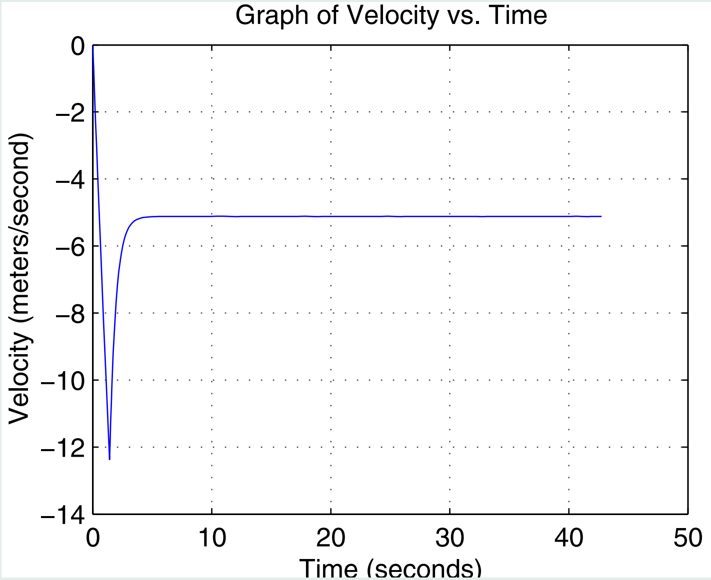

Figure 6: The graph shows the velocity getting arbitrarily close to terminal velocity very early in the jump

paratroopers must not only have enough time to decelerate, but also have enough time to determine if their parachute has deployed properly and take corrective action if it has not. This means that they must be able to deploy their reserve parachute.

It takes approximately six seconds for a jumper to realize that their parachute has not deployed and manually deploy their reserve [3]. Due to this fact, our new time of deployment (t_{d2}) will be equal to $t_d + 6\,\text{s}$ which is equal to $7.3626\,\text{s}$. This allows the jumper to reach the end of the static line, and then six seconds to react to the fact that the primary parachute has not opened. At t_{d2} the reserve parachute will deploy, and our new coefficient of friction will be k_3.

The reserve parachute is smaller, with less surface area, and therefore has a lower coefficient of drag. It therefore does not decelerate the paratrooper as much. The maximum allowable impact speed for the reserve parachute should not exceed $-5.3\,\text{m/s}$ [3]. This may not prevent injury, but can save the paratroopers life. The purpose of the reserve parachute is not to replace the primary, but simply to save the jumper's life in the event of a primary parachute malfunction.

To adjust for this necessary safety margin we must reevaluate the model incorporating these new parameters. The first step is to use these new parameters in our model to calculate t_{i2}. Figure 4 shows that the velocity passes through $-5.3\,\text{m/s}$ on the way to terminal velocity. A quick check using a numerical solver shows that the velocity is equal to $-5.3\,\text{m/s}$ at $t_{i2} = 11.3739\,\text{s}$. Now we know the value of t_{i2}, which is the time at which we reach our optimal position. If we plug our value for t back into Equation (3)and get a position of $-224.46\,\text{m}$. If we take the absolute value of this position we will have our optimal height for releasing paratroopers, $224.46\,\text{m}$

In practice, this is also lower than the military usually jumps. During basic parachuting school, soldiers typically jump from about $545\,\text{m}$. This gives beginners plenty of time to acclimatize to this new environment, as well as a greater margin of error for assessing whether or not the primary parachute has deployed and to act accordingly. During training jumps in peace time, even with seasoned veterans, the typical release height is approximately 275 - 320 m. This also gives veteran jumpers a slight margin of error for deploying their reserve parachute in the event of a primary parachute malfunction. During combat, when there is less allowable tolerance for error, my conclusion of 224.46 m seems to be reasonably accurate.

In order to visualize the results, it is best to see a graphical representation. To do this I must re-run the numerical routine, with the initial conditions $x(0) = 224.46$, and $v(0) = 0$. On the graph of position vs. time (Figure 5), it easy to see that if the jumper is released at our optimal height of 224.46 m they will impact with the ground at approximately $t = 42.7525$ s. Furthermore, the graph of velocity vs. time (Figure 6) shows that the jumper will get arbitrarily close to terminal velocity early in the jump, giving them plenty of time to prepare for landing.

3. Analysis of this Model

Although this model can make reasonably accurate predictions about the behavior of a falling paratrooper, I feel that it is appropriate to address its limitations. As with any mathematical model of a real life event, it was necessary to simplify these events in order to make the calculations manageable (Especially for an introductory DE course). A parachute fall is comprised of an extremely complex series of relationships. It would be virtually impossible to account for all of the factors involved, such as wind shear and the porosity of the parachute. I believe that a few of these are especially worth noting. First, as I previously noted, I considered motion only in the vertical direction. The forward motion of the jumper, caused by velocity imparted by the plane, was ignored for the sake of simplicity. This is a standard assumption in a basic Ordinary Differential Equations treatment of the parachute problem. Second, the piece-wise constant function for k represents a big oversimplification. It implies that the transition from free-fall to a fully open parachute is instantaneous. In reality there is a transitional period in which the actual values of k are changing continuously as the parachute is deployed. The model for this continual change in drag during the deployment is, however, beyond the scope of this problem . Third, I chose the simple model of air resistance, throughout the jump, in which resistance is proportional to velocity. It has been determined that in many cases that air resistance proportional to the square of the velocity, during certain stages of the jump, would more correctly model reality [2].

I is my belief that even with these simplifications this is an effective model. It provides an effective tool for students in an introductory differential equations course to understand the motion of a falling paratrooper, model that motion, and to make reasonable predictions as to the optimal height to release paratroopers.

4. Works Cited

References

[1] Arnold, Dave *College of the Redwoods*

[2] Meade, Douglas B.,Allan A. Struthers *Differential Equations in the New Millennium: the Parachute Problem*

[3] Meade, Douglas B. *ODE Models for the Parachute Problem*

[4] Meade, Douglas B. *Maple and the Parachute Problem: Modelling with an Impact*

[5] Stanley, Michael *2/504th Parachute Infantry Regiment*

A. Event Trapping Using Matlab

In the parachute problem the coefficient of drag changes from k_1 to k_2 at t_d. This is the time at which the parachute is deployed, and occurs when the jumper reaches the end

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of the static line. Therefore, t_d occurs when $x = -9.81\,\mathrm{m}$. As previously mentioned, it is effectively impossible to solve Equation (3) for t without using a numerical solver, due to the t in the exponential. I therefore turned to Matlab.

We start by writing a function m-file describing System (1) in the usual manner.

```
function xprime=projectm(t,x,k)
xprime=zeros(2,1);
xprime(1)=x(2);
xprime(2)=-9.81-k*x(2)/97.4;
```

In order to calculate t_d, I used the event trapping capabilities of the ode45 program. The ode45 program allows you to use a pre-defined set of options to refine your calculations. One of the options available is the "Events" option. This option allows you to designate an event, defined in an event m-file. During the calculations, when the solver gets to this pre-assigned event, the solver can be instructed to stop its calculations, whereupon one can view intermediate results and calculations.

The first step in the process is to set the "Event" property in an options structure. This is easily accomplished with Matlab's `odeset` command.

```
options=odeset('Events',@eventsm);
```

I have defined the event to be trapped in the function m-file `eventsm`.

```
function [value,isterminal,direction] = eventsm(t,x,k)
value=x(1)+9.81;
isterminal = 1;
direction = 0;
```

Here `value` is the event to be trapped. It must be in the format of a function equal to zero. The event(s) are defined as the zero crossings of this function. Since, in this case, I want to find out when the position ($x(1)$) is equal to -9.81 m, I format my event value as `x(1)+9.81`. If `isterminal` is equal to 0, integration will continue after after hitting an event. Because I wanted integration to stop, I entered a 1. The `direction` is the desired direction of the zero crossings. Set `direction` equal to -1 for crossings in the negative direction only, and equal to 1 for crossings in the positive direction only. I set `direction` equal to 0 in order to get all crossings (because I did not want to think too hard).

The next step is to run the ode45 program with the output assigned to vectors for later use.

```
[t,x,te,xe,ie]=ode45(@projectm,[tstart,tfinal],x0,options,k1);
```

Here `te` is a vector containing the time(s) of the event(s), `xe` is a vector containing the values of position and velocity at the time(s) of the event(s), `ie` is an indices vector specifying which events occurred (not relevant to this problem), `options` are the options defined above, and `k1` is the value of k_1.

When the ode45 program is run, it automatically stops when it reaches the position $x = -9.81$ m. At this time I instructed ode45 to run again using the following command

```
[t,x,te,xe,ie]=ode45(@projectm,[t2start,tfinal],x0,options,k2);
```

Here `t2start` is the time of the event, `x0` is a vector containing the values of the position and velocity at the time of the event, `options` are the same options defined above, and `k2` is the value of k_2.

I accumulated the time output of the two calculations into one vector, and the position and velocity output into another. This data can then be used to determine t_i.

- You may view my Matlab m-file by clicking on this link.

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- You can get more help on the use and syntax of ode45, options, or event trapping by typing the following at the Matlab prompt.

```
>> help ode45
>> help odeset
>> help ballode
```