The Game of Quatrainment Revisited

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Abstract

Quatrainment is a simple computer game by today's standards of computer graphics, sound, T1 connections, and multiple user domains. Yet it remains a challenging puzzle. In this article we will examine the game and its rules. Then we will develop a few mathematical tools based on linear algebra. These tools will enable us to easily and quickly solve the game.

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1. The Game of Quatrainment

The game of Quatrainment is a game that lends itself readily to computer technology. There have been several versions of this game written to run on various platforms, the most recent of which is a version written specifically for this article by Jeremy Miller to run on the Windows 95 and NT platforms.

The game consists of an initial 4×4 array with a random pattern of x's and a target 4×4 array with a random pattern of x's. Each cell of the 4×4 array has two possible states, it either contains an x or it is empty. The object of the game is to convert the initial array to the target array by a series of "moves" (we'll call them inputs) in a short period of time using a minimal number of inputs.

To "move", the player clicks in any cell of the array. The result of the input depends on whether the input is a corner, edge, or center cell. In the following illustrations, the first picture shows the initial state, the second picture shows the input cell, and the last picture shows the result of that input.

If a player selects a corner cell, the states of the six cells in that corner are changed as shown in Figure 1 on page 3.

If a player selects an edge cell, the states of the three cells that border on that cell are changed as shown in Figure 2 on page 3.

If the player selects a center cell, its state and that of the four cells that border it are changed as shown in Figure 3 on page 4.

Sometimes the solution will be evident. As an example, see Figure 4 on page 4.

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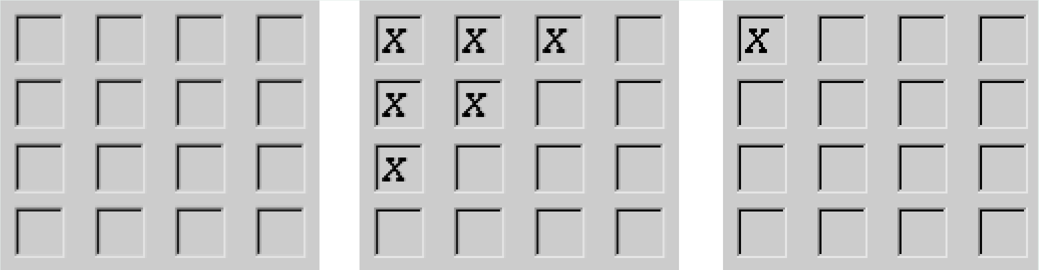

Figure 1: To go from the state on the left to the state in the middle, input the cell shown on the right.

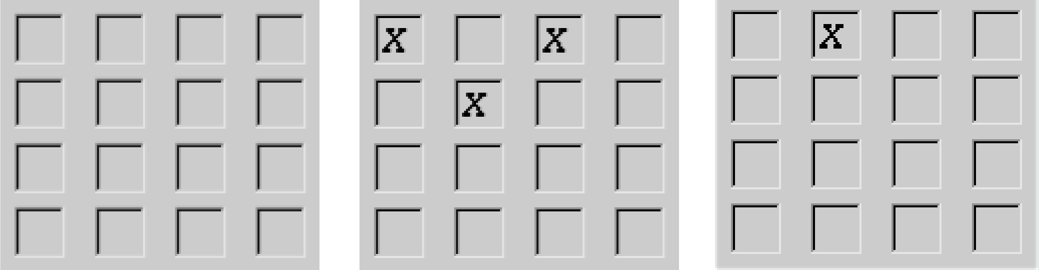

Figure 2: To go from the state on the left to the state in the middle, input the cell shown on the right.

But it's not always that simple. For instance, in Figure 5 on page 5 the solution is not immediately evident. The solution requires eleven "moves".

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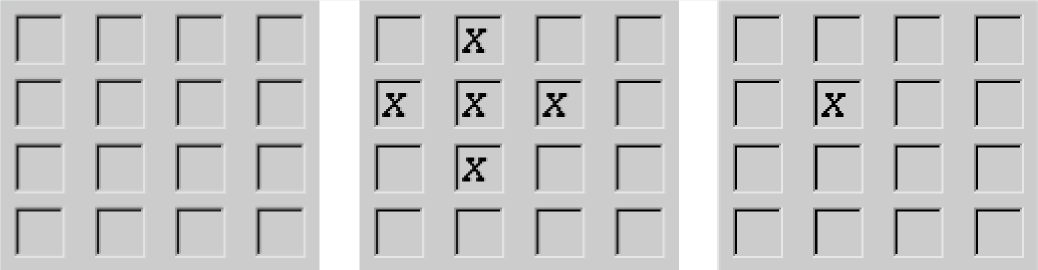
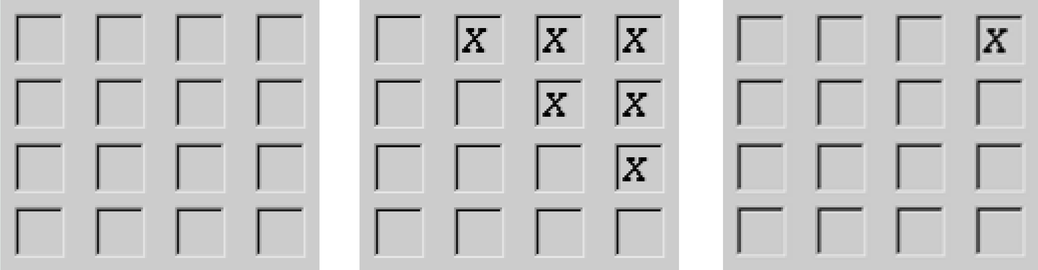


	X		
X	X	X	
	X		

	X		

Figure 3: To go from the state on the left to the state in the middle, input the cell shown on the right.



	X	X	X
		X	X
			X

			X

Figure 4: To go from the state on the left to the state in the middle, input the cell shown on the right.

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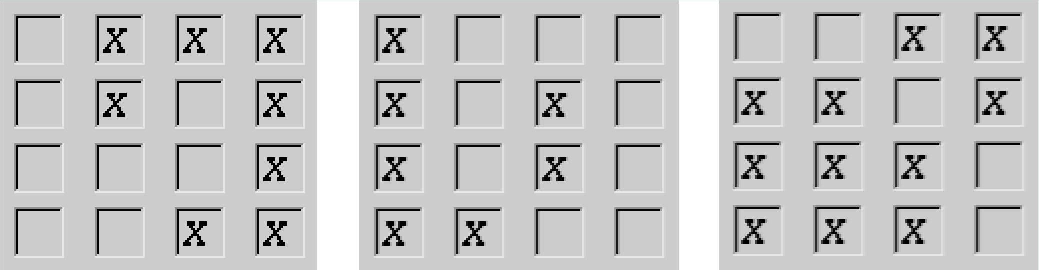

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Figure 5: To go from the state on the left to the state in the middle, input the cells shown on the right.

2. Toward a Solution

Is there a way to find (mathematically) a general solution that will enable us to find the shortest sequence of inputs to get from the initial state to the target state? Happily, the answer is yes. We will start with some matrix representations of the inputs, define them as a basis for all 4 × 4 matrices with ones or zeros as entries, then look at some solution algorithms.

First, we must define a field, and a vector space. Let $\mathbb{Z}_2$ be a field containing only zero and one. In this field, scalar multiplication is defined in the usual manner. Addition is defined in the usual manner as well, but the reader is cautioned that $1 + 1 = 0$ since there are only ones and zeros in the field. $\mathbb{Z}_2^{4\times4}$ is the set of all 4×4 matrices with ones or zeros as entries. $\mathbb{Z}_2^{4\times4}$ is a vector space over $\mathbb{Z}_2$ since:

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1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.
2. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$
3. There exists an element $\mathbf{0}$ such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$
4. For each $\mathbf{x}$ there exists an element $-\mathbf{x}$ such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$.
5. $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$.
6. $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$
7. $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$
8. $1 \cdot \mathbf{x} = \mathbf{x}$
9. If $\mathbf{x} \in \mathcal{V}$ and α is a scalar, then $\alpha\mathbf{x} \in \mathcal{V}$
10. If $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ and α is a scalar, then $\alpha\mathbf{x} + \mathbf{y} \in \mathcal{V}$

Notice that each element of $\mathbb{Z}_2^{4\times 4}$ is its own additive inverse, since $1 + 1 = 0$.

Let's take a look at the possible Quatrainment inputs in the light of the field and vector space that we just defined. If we let a one represent an x, and a zero an empty cell, we can look at Quatrainment mathematically. The entire game takes place in $\mathbb{Z}_2^{4\times 4}$. The sixteen matrices shown below form a

basis for $\mathbb{Z}_2^{4\times 4}$, so the dimension of $\mathbb{Z}_2^{4\times 4}$ is sixteen.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The sixteen matrices shown below represent all possible Quatrainment inputs. Do they also form a basis for $\mathbb{Z}_2^{4\times 4}$. In other words, is it possible to express any matrix with ones or zeros as entries using a linear combination of these inputs? Unless this is possible, Quatrainment will not always have a solution. Let's name them for the sake of convenience.

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$$M_0 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M_5 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_6 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M_7 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M_8 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$M_9 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad M_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad M_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad M_{13} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad M_{14} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$M_{15} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

At this point it will be convenient to introduce a shorthand for the en-

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tries of these matrices as well. We will define the entries of M_i where $i = (0, 1, 2, 3, , , , 15)$ thus:

$$\begin{bmatrix} M_{i,0} & M_{i,1} & M_{i,2} & M_{i,3} \\ M_{i,4} & M_{i,5} & M_{i,6} & M_{i,7} \\ M_{i,8} & M_{i,9} & M_{i,10} & M_{i,11} \\ M_{i,12} & M_{i,13} & M_{i,14} & M_{i,15} \end{bmatrix}$$

Recall that for the $M_i's$ to be linearly independent, the equation $a_0M_0 + a_1M_1 + a_2M_2 + \cdots + a_{15}M_{15} = 0$ can have only the trivial solution $\mathbf{a} = \mathbf{0}$. Expanding this, we get the following system of 16 equations in sixteen variables.

$$\begin{aligned}
a_0M_{0,0} + a_1M_{1,0} + a_2M_{2,0} + \cdots + a_{15}M_{15,0} &= 0 \\
a_0M_{0,1} + a_1M_{1,1} + a_2M_{2,1} + \cdots + a_{15}M_{15,1} &= 0 \\
a_0M_{0,2} + a_1M_{1,2} + a_2M_{2,2} + \cdots + a_{15}M_{15,2} &= 0 \\
a_0M_{0,3} + a_1M_{1,3} + a_2M_{2,3} + \cdots + a_{15}M_{15,3} &= 0 \\
a_0M_{0,4} + a_1M_{1,4} + a_2M_{2,4} + \cdots + a_{15}M_{15,4} &= 0 \\
a_0M_{0,5} + a_1M_{1,5} + a_2M_{2,5} + \cdots + a_{15}M_{15,5} &= 0 \\
a_0M_{0,6} + a_1M_{1,6} + a_2M_{2,6} + \cdots + a_{15}M_{15,6} &= 0 \\
a_0M_{0,7} + a_1M_{1,7} + a_2M_{2,7} + \cdots + a_{15}M_{15,7} &= 0 \\
a_0M_{0,8} + a_1M_{1,8} + a_2M_{2,8} + \cdots + a_{15}M_{15,8} &= 0
\end{aligned}$$

$$
\begin{aligned}
a_0 M_{0,9} + a_1 M_{1,9} + a_2 M_{2,9} + \cdots + a_{15} M_{15,9} &= 0 \\
a_0 M_{0,10} + a_1 M_{1,10} + a_2 M_{2,10} + \cdots + a_{15} M_{15,10} &= 0 \\
a_0 M_{0,11} + a_1 M_{1,11} + a_2 M_{2,11} + \cdots + a_{15} M_{15,11} &= 0 \\
a_0 M_{0,12} + a_1 M_{1,12} + a_2 M_{2,12} + \cdots + a_{15} M_{15,12} &= 0 \\
a_0 M_{0,13} + a_1 M_{1,13} + a_2 M_{2,13} + \cdots + a_{15} M_{15,13} &= 0 \\
a_0 M_{0,14} + a_1 M_{1,14} + a_2 M_{2,14} + \cdots + a_{15} M_{15,14} &= 0 \\
a_0 M_{0,15} + a_1 M_{1,15} + a_2 M_{2,15} + \cdots + a_{15} M_{15,15} &= 0
\end{aligned}
$$

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Which leads to the augmented matrix:

$$\begin{bmatrix}
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}$$

or, in reduced row echelon form:

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix},$$

indicating that $\mathbf{a} = \mathbf{0}$ and the matrices $M_0, M_1, M_2, \ldots, M_{15}$ are linearly independent. Since they are linearly independent and there are sixteen of them, they form a basis for $\mathbb{Z}_2^{4\times 4}$. Since they form a basis for $\mathbb{Z}_2^{4\times 4}$, any 4×4 matrix with only ones or zeros as entries can be expressed as a linear combination of these basis matrices. Consequently, the game will always have a solution.

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3. A Solution Algorithm

Since these sixteen matrices form a basis for the subspace, and each matrix is its own additive inverse, any input array can be transformed into any target array by applying, at most, sixteen input selections, but how do we find the right selections?

We could look at each cell of the input and target arrays to see which cells need to be changed, find some combination of inputs that will transform each corner, edge, and center cell, apply that combination of inputs to each cell that needs to be changed, count how many times each basis matrix was used, find that number's equivalence class in $\mathbb{Z}_2$, and apply the necessary inputs to do the transformation. Unfortunately, Quatrainment keeps track of the elapsed time as well as the number of moves. We need something a lot faster than that.

Ideally, we could define a 4×4 matrix with a one in each cell representing an input needed to transform the initial array into the target array. We've already done most of the work, so the answer is fairly simple. First, notice

that the columns of $Z =$

$$\begin{bmatrix}
1&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&1&0&1&0&0&0&0&0&0&0&0&0&0\\
1&1&0&1&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&0&0&1&0&0&0&0&0&0&0&0\\
1&0&0&0&0&1&0&0&1&0&0&0&1&0&0&0\\
1&1&0&0&1&1&1&0&0&1&0&0&0&0&0&0\\
0&0&1&1&0&1&1&1&0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&1&0&0&0&0&1&0&0&0&1\\
1&0&0&0&1&0&0&0&0&1&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0&1&1&1&0&1&1&0&0\\
0&0&0&0&0&0&1&0&0&1&1&1&0&0&1&1\\
0&0&0&1&0&0&1&1&0&0&1&0&0&0&0&1\\
0&0&0&0&0&0&0&0&1&0&0&0&1&1&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&1&0&1&1\\
0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&1
\end{bmatrix}$$

are our basis matrices transposed with the columns stacked on top of each other (column 1 is $[M_0$ row 1, M_0 row 2, M_0 row 3, M_0 row $4]^T$, column 2 is $[M_1$ row 1, M_1 row 2, M_1 row 3, M_1 row $4]^T$, etc). Since Z has linearly independent columns, it is invertible.

The only problem is that Z^{-1} has rational entries, specifically, multiples of 9^{-1}. But 9^{-1} is simply the number we need to multiply by 9 to get one. In $\mathbb{Z}_2$, the equivalence class of 9 is 1. The number we need to multiply 1 by

to get 1 is 1. So 9^{-1} is really 1 in $\mathbb{Z}_2$ By the same logic, the numerators that are not either a one or a zero belong to either the equivalence class 1 or 0 in $\mathbb{Z}_2$.

If we then take the input array and add it in $\mathbb{Z}_2$ to the target array, we get an array with an 1, or an x, if you will, in each cell that needs to be changed. We'll call this matrix C. Now we have all the elements of a solution.

If we take C, transpose it, and stack the columns on top of each other, we get a column vector that we can multiply on the right with Z^{-1}. We want to do this because we are really taking each cell that needs to be changed and multiplying it by the inverse of the matrix representation of the result of inputting that cell.

$$\begin{aligned} Z\mathbf{a} &= \mathbf{c} \\ Z^{-1}Z\mathbf{a} &= Z^{-1}\mathbf{c} \\ \mathbf{a} &= Z^{-1}\mathbf{c} \end{aligned}$$

We take the resulting column vector, here we called it **a**, turn it back into a 4×4 matrix by unstacking the columns, transpose it, and we have our solution matrix. It has a 1 (x) in each cell that needs to be input to win the game. This is the solution algorithm that our program uses when the player clicks the solve button.

References

[1] Leon, Steven J., *Linear Algebra with Applications*, Prentice Hall, 1998

[2] Gantner, T. E., The Game of Quatrainment, *Mathematics Magazine*, 61.1, 1988, 29-34

[3] Miller, Jeremy and Ken Menkveld, *Quatrainment Ver. 2.0*, Software, November, 2000, `http://online.redwoods.cc.ca.us/instruct/darnold/laproj/fall2000/kenjer/quatrainment.zip`

[4] Arnold, David, *Modular Arithmetic and Hill Ciphers in Matlab*, 1996, `http://online.redwoods.cc.ca.us/instruct/darnold/LinAlg/activity.htm`

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