Projecting Into Fourier

The Sauceman and The Professor

December 10th, 2001

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Abstract

This paper shows how a Fourier approximation for a function is really the projection of a function onto the vector space spanned by the orthonormal set

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos x, \frac{1}{\sqrt{\pi}}\sin x, \frac{1}{\sqrt{\pi}}\cos 2x, \frac{1}{\sqrt{\pi}}\sin 2x, \frac{1}{\sqrt{\pi}}\cos 3x, \frac{1}{\sqrt{\pi}}\sin 3x \cdots \right\}.$$

Introduction

In 1807, Joseph Fourier proposed that any function can be expressed as a series of sinusoidal curves. His proposal was rejected by the mathematical community of the time because of the lack of proof of his hypothesis.

As it turns out, Fourier was right and caused mathematicians to rethink the definition of a function. His theory introduced the topic of harmonic analysis or Fourier Analysis. The applications of Fourier Analysis has since spread beyond the borders of pure mathematics and has proven valuable in chemistry, physics, life sciences, and beyond.

Mathematically, the Fourier series is a projection of a function into a special space spanned by a set of orthogonal, sinusoidal functions.

Vector Spaces and Inner Product Spaces

A vector with n components lives in the vector space $\mathbb{R}^n$, and usually that is the space we deal with in linear algebra, with different values of n, such as $\mathbb{R}^3$, $\mathbb{R}^4$, etc.

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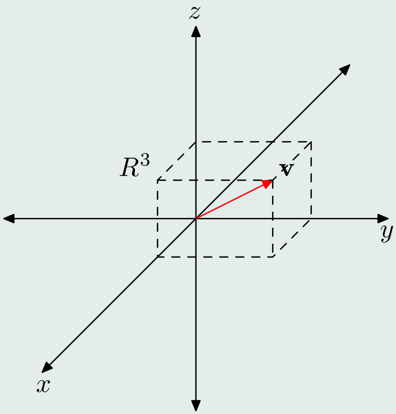

Figure 1: The Vector Space $\mathbb{R}^3$.

However, all continuous functions also live in a vector space. On the interval from a to b, that space is notated as C[a,b]. It would be easy to show that the ten properties of a vector space are satisfied by this space using the fact that the sum of any continuous functions is a continuous function and the product of a continuous function and a constant is a continuous function.

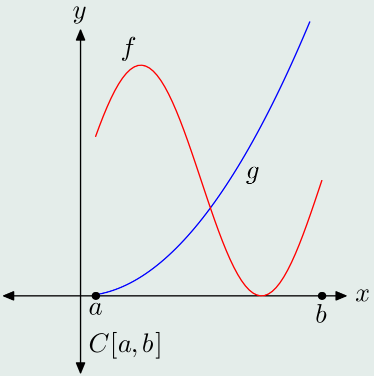

Figure 2: The Space of Continuous Function $C[a,b]$.

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For any vectors f and g in C[a,b], the inner product is notated as $< f, g >$ and is defined to be $\int_a^b f(x)g(x)dx$. For C[a,b] to to be an inner product space, in addition to satisfying the ten properties of a vector space, $< f, g >$ must satisfy the following conditions of inner products:

1. $< f, f > \quad \geq 0$
2. $< f, f >= 0$ if and only if $f = 0$
3. $< f, g >=< g, f >$
4. $< \alpha f, g >=< f, \alpha g >= \alpha < f, g >$
5. $< f, g + h >=< f, g > + < f, h >$

1. All of these conditions are easy to show. For the first one, we make use of the fact that the square of any real number is greater than or equal to zero.

$$\begin{aligned} < f, f > &= \int_a^b f(x)f(x)dx \\ &= \int_a^b (f(x))^2 dx \\ &\geq 0 \end{aligned}$$

2. Since $\int_a^b (f(x))^2 dx$ can only equal zero if $f(x)$ equals zero,

$$< f, f >= 0 \quad \text{if and only if} \quad f(x) = 0.$$

3. Multiplication of functions is commutative, so we can rearrange the order of $f(x)$ and $g(x)$.

$$\begin{aligned} < f, g > &= \int_a^b f(x)g(x)dx \\ &= \int_a^b g(x)f(x)dx \\ &=< g, f > . \end{aligned}$$

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4. Since α is a constant, it can be moved around and even pulled out of the integral.

$$\begin{aligned} < \alpha f, g > &= \int_a^b (\alpha f(x) g(x) dx \\ &= \int_a^b (f(x) \alpha g(x) dx \\ &= < f, \alpha g > \end{aligned}$$

   Or,

$$\begin{aligned} \int_a^b \alpha f(x) g(x) dx &= \alpha \int_a^b f(x) g(x) dx \\ &= \alpha < f, g > . \end{aligned}$$

5. This one simply follows from the distributive property and the fact that the integral of a sum is the sum of the integrals.

$$\begin{aligned} < f, g + h > &= \int_a^b (f(x)(g(x) + h(x)) dx \\ &= \int_a^b (f(x) g(x) + f(x) h(x)) dx \\ &= \int_a^b f(x) g(x) dx + \int_a^b f(x) h(x) dx \\ &= < f, g > + < f, h > \end{aligned}$$

Now that we have established that the space of continuous function, $C[a, b]$, is an inner product space, We can calculate the magnitude of a function by taking the inner product of that function with itself.

$$\begin{aligned} \|f\|^2 &= < f, f > \\ &= \int_a^b f(x) f(x) dx \\ &= \int_a^b (f(x))^2 dx \end{aligned}$$


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The Orthogonal Set

To begin with, consider the set of vectors $\{1, \cos x, \sin x, cos2x, \sin 2x, \cos 3x, \cdots\}$ in the vector space C[0,2π]. If every one of these vectors is perpendicular to every other one on $C[0, 2\pi]$, then they form a mutually orthogonal set on that space.

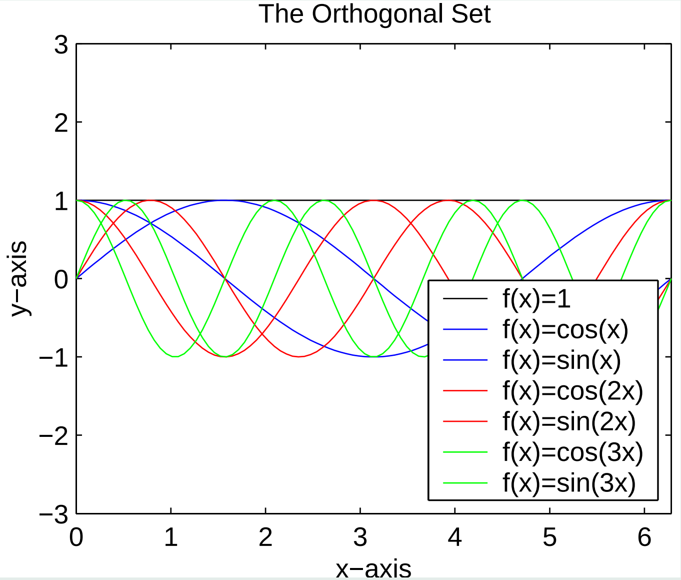

Figure 3: The set of orthogonal functions.

They certainly don't look orthogonal, at least not orthogonal as we are used to. However, if the inner product of any two of any the vectors is zero, the set is indeed orthogonal. To show that this is true, we need to prove the following:

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$$
\begin{aligned}
<\sin nx, \sin mx> &= 0, \quad \text{for} \quad n \neq m && (1)\\
<\sin nx, \cos mx> &= 0 && (2)\\
<\cos nx, \cos mx> &= 0, \quad \text{for} \quad n \neq m && (3)
\end{aligned}
$$

1. In order to show $<\sin nx, \sin mx>= 0$, for $n \neq m$, we take the inner product of the two elements.

$$<\sin nx, \sin mx> = \int_0^{2\pi} \sin nx \sin mx dx$$

Using the sine product identity, $\sin A \cos B = 1/2[\cos(A-B) - \cos(A+B)]$, we get

$$
\begin{aligned}
<\sin nx, \sin mx> &= \frac{1}{2}\int_0^{2\pi} [\cos(nx - mx) - \cos(nx + mx)]dx\\
&= \frac{1}{2}\int_0^{2\pi} \cos[x(n-m)]dx - \frac{1}{2}\int_0^{2\pi} \cos[x(n+m)]dx\\
&= \frac{1}{2}\left[\frac{1}{n-m}\sin[x(n-m)]\right]_0^{2\pi} - \frac{1}{2}\left[\frac{1}{n+m}\sin[x(n+m)]\right]_0^{2\pi}\\
&= \frac{1}{2}\left[\frac{1}{n-m}[\sin[2\pi(n-m)] - \sin[0(n-m)]]\right.\\
&\qquad \left. - \frac{1}{n+m}[\sin[2\pi(n+m)] - \sin[0(n+m)]]\right].
\end{aligned}
$$

Any multiple of 2π in the sine function will produces 0. Since m and n are both integer values, the following results.

$$
\begin{aligned}
<\sin nx, \sin mx> &= \frac{1}{2}\left[\frac{1}{n-m}[0-0] - \frac{1}{n+m}[0-0]\right]\\
&= \frac{1}{2}\left[\frac{1}{n-m}[0] - \frac{1}{n+m}[0]\right]\\
&= \frac{1}{2}[0-0]\\
&= 0
\end{aligned}
$$

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2. In order to show that $< \sin nx, \cos mx >= 0$, we evaluate the inner product.

$$< \sin nx, \cos mx > = \int_0^{2\pi} \sin nx \cos mx \, dx$$

Using the identity $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$,

$$\begin{aligned}
< \sin nx, \cos mx > &= \int_0^{2\pi} \frac{1}{2}[\sin(nx-mx) + \sin(nx+mx)]dx \\
&= \frac{1}{2}\int_0^{2\pi} \sin((n-m)x)dx + \frac{1}{2}\int_0^{2\pi} \sin((n+m)x)dx \\
&= \frac{1}{2}\left[-\frac{1}{n-m}\cos((n-m)x)dx\right]_0^{2\pi} + \frac{1}{2}\left[-\frac{1}{n+m}\cos((n+m)x)dx\right]_0^{2\pi} \\
&= -\frac{1}{2}\left(\frac{1}{n-m}\right)[\cos((n-m)2\pi) - \cos((n-m)0)] \\
&\qquad -\frac{1}{2}\left(\frac{1}{n+m}\right)[\cos((n+m)2\pi) - \cos((n+m)0)]
\end{aligned}$$

Since the cosine of any integer multiple of 2π equals 1, and n and m are integers,

$$\begin{aligned}
< \sin nx, \cos mx > &= -\frac{1}{2}\left(\frac{1}{n-m}\right)[1-1] - \frac{1}{2}\left(\frac{1}{n+m}\right)[1-1] \\
&= 0.
\end{aligned}$$

3. As before, to show that $< \cos nx, \cos mx >= 0$ for $n \neq m$, we show that the inner product.

$$< \cos nx, \cos mx > = \int_0^{2\pi} \cos nx \cos mx dx$$

Using the trigonometric product identity for cosine, $\cos A \cos B = 1/2[\cos(A-B) + \cos(A+B)]$, we get

$$\begin{aligned}
<\cos nx, \cos mx> &= \frac{1}{2}\int_0^{2\pi} [\cos(nx - mx) + \cos(nx + mx)]dx \\
&= \frac{1}{2}\int_0^{2\pi} [\cos[x(n-m)] + \cos[x(n+m)]]dx \\
&= \frac{1}{2}\int_0^{2\pi} \cos[x(n-m)]dx + \frac{1}{2}\int_0^{2\pi} \cos[x(n+m)]dx \\
&= \frac{1}{2}\left[\frac{1}{n-m}\sin[x(n-m)]\right]_0^{2\pi} + \frac{1}{2}\left[\frac{1}{n+m}\sin[x(n+m)]dx\right]_0^{2\pi} \\
&= \frac{1}{2}\left[\frac{1}{n-m}[\sin[2\pi(n-m)] - \sin[0(n-m)]]\right. \\
&\qquad \left. + \frac{1}{n+m}[\sin[2\pi(n+m)] - \sin[0(n+m)]]\right]
\end{aligned}$$

Since any multiple of 2π or 0 within sine will produce 0, and m and n are interger values,

$$\begin{aligned}
<\cos nx, \cos mx> &= \frac{1}{2}\left[\frac{1}{n-m}[0-0] + \frac{1}{n+m}[0-0]\right] \\
&= 0
\end{aligned}$$

Note that by choosing values of n and m, the inner product of any two terms in our set can be calculated by either inner product (1), (2), or (3). Since we have proven that all three of these inner products equal zero for arbitrary integer valuess of n and m, the set is mutually orthogonal.

From Orthogonal to Orthonormal

Now orthogonal is good, but as any linear algebra student knows, orthonormal is better. If we take every vector from our set and divide by its length, we will then have an orthonormal set. To do that, we will first calculate the squared length of each vector.

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1. In order to find the magnitude of the sine terms, we find that $\|\sin nx\|^2 =< \sin nx, \sin nx >$ for $n > 0$. From this we evaluate the integral to find the squared magnitude.

$$\|\sin nx\|^2 =< \sin nx, \sin nx > = \int_0^{2\pi} (\sin nx)^2 dx$$

Using the half angle identity,

$$\begin{aligned} \|\sin nx\|^2 &= \int_0^{2\pi} \frac{1}{2}(1-\cos 2nx)dx \\ &= \frac{1}{2}\int_0^{2\pi}(1-\cos 2nx)dx \\ &= \frac{1}{2}\left[x - \frac{1}{2}\sin 2nx\right]_0^{2\pi} \\ &= \frac{1}{2}\left[\left(2\pi - \frac{1}{2}\sin 2n(2\pi)\right) - \left(0 - \frac{1}{2}\sin 2n(0)\right)\right] \end{aligned}$$

Since n is an integer, and any multiple of 2π within sine will produce 0.

$$\begin{aligned} \|\sin nx\|^2 &= \frac{1}{2}\left[\left(2\pi - \frac{1}{2}(0)\right) - \left(0 - \frac{1}{2}(0)\right)\right] \\ &= \frac{1}{2}(2\pi) \\ &= \pi \end{aligned}$$

Thus, the magnitude of any of the sine terms is $\sqrt{\pi}$.

2. In a similar fashion, the computation to find the magnitude of the cosine terms, for $n > 0$ are given by the inner product $\|\cos nx\|^2 =< \cos nx, \cos nx >$. Evaluating the inner product,

$$\|\cos nx\|^2 =< \cos nx, \cos nx > = \int_0^{2\pi} (\cos nx)^2 dx,$$

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Using the double angle identity,

$$
\begin{aligned}
\|\cos nx\|^2 &= \int_0^{2\pi} \frac{1}{2}(1+\cos 2nx)dx \\
&= \frac{1}{2}\int_0^{2\pi} (1+\cos 2nx)dx \\
&= \frac{1}{2}\left[x + \frac{1}{2}\sin 2nx\right]_0^{2\pi} \\
&= \frac{1}{2}\left[\left(2\pi + \frac{1}{2}\sin 2n(2\pi)\right) - \left(0 + \frac{1}{2}\sin 2n(0)\right)\right]
\end{aligned}
$$

Since n is an integer, and any multiple of 2π within sine will produce 0.

$$
\begin{aligned}
\|\cos nx\|^2 &= \frac{1}{2}\left[\left(2\pi + \frac{1}{2}(0)\right) - \left(0 + \frac{1}{2}(0)\right)\right] \\
&= \frac{1}{2}(2\pi) \\
&= \pi
\end{aligned}
$$

Thus the magnitude of all the cosine terms with $n > 0$ is $\sqrt{\pi}$, just like the sine terms.

3. In order to find the magnitude of the first terms, $\cos 0x$ The only cosine term for which the magnitude is not $\sqrt{\pi}$ is with $n = 0$.

$$
\begin{aligned}
\|\cos 0x\|^2 =< \cos 0x, \cos 0x >=< 1, 1 > &= \int_0^{2\pi} (1)^2 dx \\
&= \int_0^{2\pi} dx \\
&= x|_0^{2\pi} \\
&= 2\pi.
\end{aligned}
$$

The magnitude of the first term, $\cos 0x = 1$ is then $\sqrt{2\pi}$.

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The Orthonormal Set

Now to make our orthogonal set orthonormal, we just divide by these magnitudes and get

$$\left\{ \frac{1}{\sqrt{2\pi}},\ \frac{1}{\sqrt{\pi}}\cos x,\ \frac{1}{\sqrt{\pi}}\sin x,\ \frac{1}{\sqrt{\pi}}\cos 2x,\ \frac{1}{\sqrt{\pi}}\sin 2x,\ \frac{1}{\sqrt{\pi}}\cos 3x,\ \cdots \right\}$$

.

Projecting a Function onto the Space Spanned by Our Orthogonal Set

When a vector lives outside a vector space, what vector within the space is closest to the outside vector? The answer to this is the projection of that vector onto the space spanned by the vectors. This is represented by the following image.

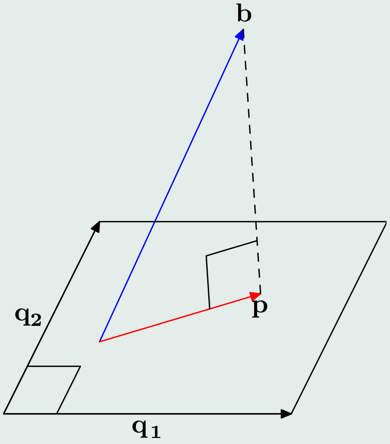
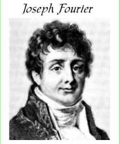


Figure 4: The projection of b onto the space spanned by q_1 and q_2.

If we form a matrix of independent vectors, then we can project the outside vector onto the space spanned by the columns. However, as a known fact, *orthogonal is good* and **orthonormal is better**.



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Suppose that we have a $n \times n$ matrix Q composed of orthonormal vectors. Using the idea of least squares approximation, the formula for the projection ($\mathbf{p}$) of a vector onto the space spanned by the columns of Q is

$$\mathbf{p} = Q(Q^TQ)^{-1}Q^T\mathbf{b}.$$

Since Q is an orthogonal matrix, $(Q^TQ) = I$. Thus,

$$\begin{aligned}\mathbf{p} &= QI^{-1}Q^T\mathbf{b} \\ &= QQ^T\mathbf{b}.\end{aligned}$$

Breaking the two matrices into component form, we continue with our analysis.

$$\begin{aligned}\mathbf{p} &= \begin{pmatrix}\mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 & \cdots & \mathbf{q}_n\end{pmatrix}\begin{pmatrix}\mathbf{q}_1^T \\ \mathbf{q}_2^T \\ \mathbf{q}_3^T \\ \vdots \\ \mathbf{q}_n^T\end{pmatrix}\mathbf{b}. \\ &= (\mathbf{q}_1\mathbf{q}_1^T + \mathbf{q}_2\mathbf{q}_2^T + \mathbf{q}_3\mathbf{q}_3^T + \cdots + \mathbf{q}_n\mathbf{q}_n^T)\mathbf{b} \\ &= \mathbf{q}_1\mathbf{q}_1^T\mathbf{b} + \mathbf{q}_2\mathbf{q}_2^T\mathbf{b} + \mathbf{q}_3\mathbf{q}_3^T\mathbf{b} + \cdots + \mathbf{q}_n\mathbf{q}_n^T\mathbf{b}\end{aligned}$$

We now group the innerproducts.

$$\mathbf{p} = \mathbf{q}_1(\mathbf{q}_1^T\mathbf{b}) + \mathbf{q}_2(\mathbf{q}_2^T\mathbf{b}) + \mathbf{q}_3(\mathbf{q}_3^T\mathbf{b}) + \cdots + \mathbf{q}_n(\mathbf{q}_n^T\mathbf{b})$$

Writing the innerproducts in the notation shown earlier, we get

$$\mathbf{p} =< \mathbf{b}, \mathbf{q}_1 > \mathbf{q}_1 + < \mathbf{b}, \mathbf{q}_2 > \mathbf{q}_2 + < \mathbf{b}, \mathbf{q}_3 > \mathbf{q}_3 + \ldots + < \mathbf{b}, \mathbf{q}_n > \mathbf{q}_n.$$

The vector $\mathbf{p}$ is the least square approximation of $\mathbf{b}$ in the space spanned by the columns of Q.


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In a similar fashion, we can project a function onto the space spanned by our orthonormal basis of functions. The formula for projecting a function onto our orthonormal set is

$$\mathbf{p} =< f, \frac{1}{\sqrt{2\pi}} > \frac{1}{\sqrt{2\pi}} + < f, \frac{1}{\sqrt{\pi}} \cos x > \frac{1}{\sqrt{\pi}} \cos x + < f, \frac{1}{\sqrt{\pi}} \sin x > +$$
$$\cdots + < f, \frac{1}{\sqrt{\pi}} \cos nx > \frac{1}{\sqrt{\pi}} \cos nx + < f, \frac{1}{\sqrt{\pi}} \sin nx > \frac{1}{\sqrt{\pi}} \sin nx.$$

Since innerproducts give constants, we collect the constant terms and write

$$= a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x \ldots + a_n \cos nx + b_n \sin nx.$$

Where a_k are the terms associated with cosine and b_k are the terms associated with sine. These constants are known as the Fourier Coefficients. Furthermore, the projection of f onto the space spanned by our set of functions is called a Fourier series approximation of f.

The Fourier coefficients in the Fourier Series, a_0, a_1, b_1, $\ldots$, a_n, b_n, are given by the following integrals for the space of continuous functions from 0 to 2π.

$$\begin{aligned} a_0 &=< f, \frac{1}{\sqrt{2\pi}} > \frac{1}{\sqrt{2\pi}} = \frac{1}{2\pi} \int_0^{2\pi} f(x)dx \\ a_k &=< f, \frac{1}{\sqrt{\pi}} \cos kx > \frac{1}{\sqrt{\pi}} = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx dx \\ b_k &=< f, \frac{1}{\sqrt{\pi}} \sin kx > \frac{1}{\sqrt{\pi}} = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx dx \end{aligned}$$

An Example

This idea of projecting a function onto a space is somewhat abstract. To show some applications of this idea, let us take a function that lives outside the space spanned by our orthonormal set. Consider the function $f(x) = x$:

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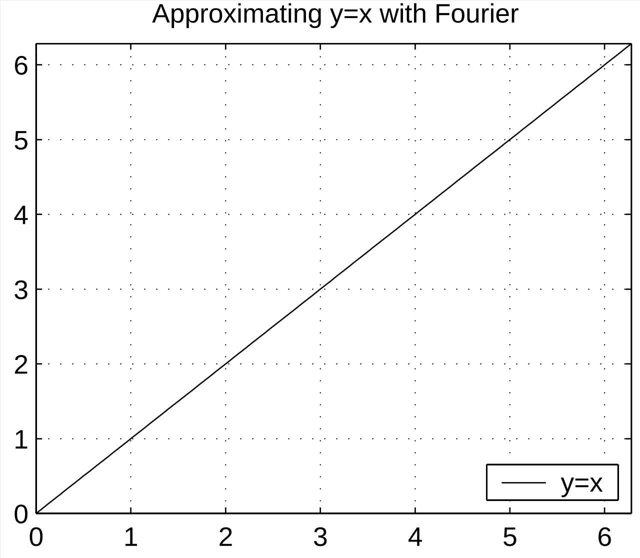

What does it mean to project the function $f(x) = x$ onto a vector space? To see graphically, let's calculate the fourier approximation for the function, which as we have shown, is the projection onto the space spanned by our orthonormal set. To begin with, we need to calculate the Fourier coefficients for $f(x) = x$,

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} x dx$$
$$a_k = \frac{1}{\pi}\int_0^{2\pi} x \cos kx dx$$
$$b_k = \frac{1}{\pi}\int_0^{2\pi} x \sin kx dx$$

The evaluation of these integrals follows.

- The a_0 Fourier coefficient, for the function $f(x) = x$, is given by evaluating the inner product

$$\begin{aligned}
a_0 &= < x, \frac{1}{\sqrt{2\pi}} > \frac{1}{\sqrt{2\pi}} = \frac{1}{2\pi}\int_0^{2\pi} x dx \\
&= \frac{1}{2\pi}\left[\frac{1}{2}x^2\right]_0^{2\pi} \\
&= \frac{1}{4\pi}(4\pi^2) \\
&= \frac{4\pi^2}{4\pi} \\
a_0 &= \pi
\end{aligned}$$

  We now have our first coefficient for the projection of the function onto the space spanned by our orthonormal set. The remaining terms are unique, and must be expressed arbitrarily for each possible k value.

- The Fourier coefficient for each cosine term is given by

$$< x, \frac{1}{\sqrt{\pi}} \cos kx > \frac{1}{\sqrt{\pi}} = \frac{1}{\pi}\int_0^{2\pi} x \cos kx dx$$

  This integral requires integration by parts. We substitute $u = x$ and $dv = \cos kx$. This gives us $du = dx$ and $v = (1/k)\sin kx$. The integral is now given by

$$= uv - \int v du.$$

  Returning to our original integral,

$$\begin{aligned}
a_k = \frac{1}{\pi}\int_0^{2\pi} x \cos kx dx &= \frac{1}{\pi}\left(\left[x\frac{1}{k}\sin kx\right]_{x=0}^{x=2\pi} - \int_0^{2\pi}\frac{1}{k}\sin kx dx\right) \\
&= \frac{1}{\pi}\left(\left[x\frac{1}{k}\sin kx\right]_0^{2\pi} - \left[-\frac{1}{k^2}\cos kx\right]_0^{2\pi}\right)
\end{aligned}$$

Any multiple of 2π or 0 within sine will produce 0. Since k is an integer value, we get

$$\begin{aligned} a_k &= \frac{1}{\pi}\Big([0-0]+[1-1]\Big) \\ &= \frac{1}{\pi}[0+0] \\ a_k &= 0 \end{aligned}$$

For all values k, the Fourier coefficients for the cosine terms will also be zero since $a_k = 0$. This means that the cosine terms will be eliminated from our projection of $f(x) = x$ onto our space.

- We now find the Fourier coefficients for the sine terms in our projection. These are found in a similar manner as before.

$$b_k =< x, \frac{1}{\sqrt{\pi}}\sin kx > \frac{1}{\sqrt{\pi}} = \frac{1}{\pi}\int_0^{2\pi} x \sin kx dx$$

Once again, we must use integration by parts to evaluate this integral. We will let $u = x$ and $dv = \sin kx$. This gives us that $du = dx$ and $v = -(1/k)\cos kx$. The integral is now given by

$$= uv - \int vdu$$

We return to our original integral.

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_0^{2\pi} x\sin kx dx = \frac{1}{\pi}\left(\left[-x\frac{1}{k}\cos kx\right]_0^{2\pi} - \int_0^{2\pi} -\frac{1}{k}\cos kx dx\right) \\ &= \frac{1}{\pi}\left(\left[-x\frac{1}{k}\cos kx\right]_0^{2\pi} - \left[-\frac{1}{k^2}\sin kx\right]_0^{2\pi}\right) \end{aligned}$$

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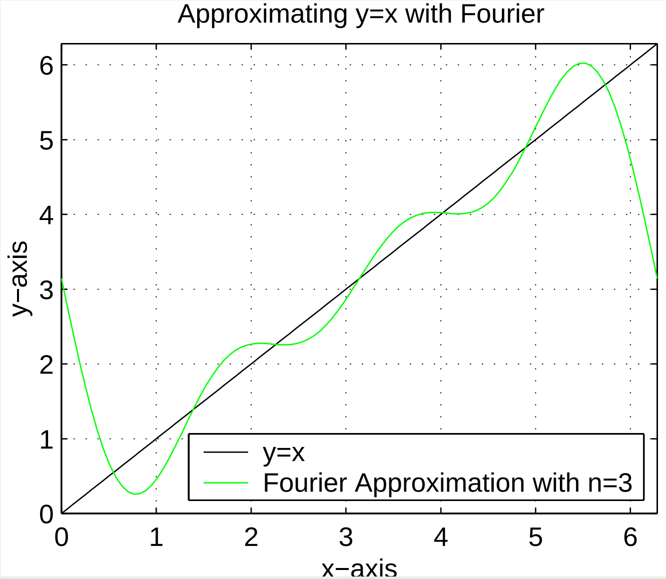
As mentioned earlier, any multiple of 0 or 2π will produce 0. Since k is an integer value, we get

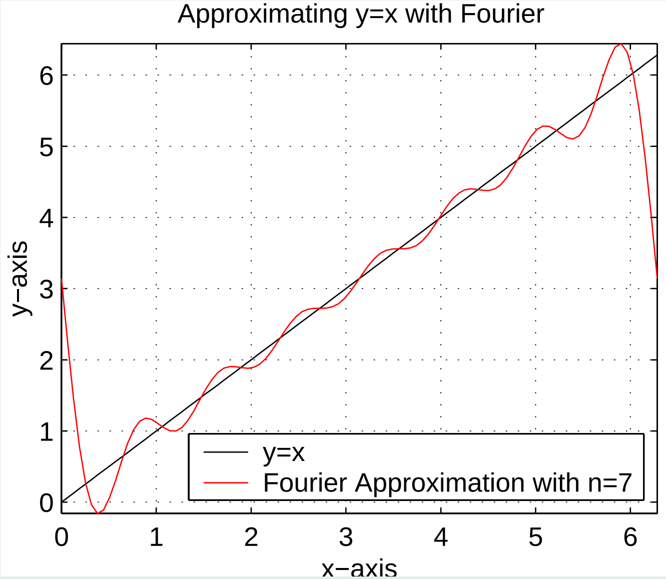
$$\begin{aligned} b_k &= \left(\left[-2\pi\left(\frac{1}{k}\right)(1)\right] + \left[-0\left(\frac{1}{k}\right)(1)\right] - \left[-\frac{1}{k^2}(0) + \frac{1}{k^2}(0)\right]\right) \\ &= \frac{1}{\pi}\left(-\frac{2\pi}{k}\right) \\ b_k &= -\frac{2}{k} \end{aligned}$$

We now have that all the Fourier coefficients associated with sine is $b_k = -2/k$.

Then the Fourier approximation becomes $p = \pi - 2\sin x - \sin 2x - \frac{2}{3}\sin 3x - \frac{1}{2}\sin 4x - \frac{2}{5}\sin 5x - \cdots$ There are an infinite number of vectors in our set, but if we project f onto the space spanned by just the first four, we get a new function that is similar to the original f.



Now if use the first 8 terms, up through $n = 7$, we get a function



If the space we project onto has all the terms up through $n = 50$ as the basis, the projection looks almost identical to f. Note that we are only dealing with the vector space $C[0, 2\pi]$. Outside of that space has nothing to do with the projection.

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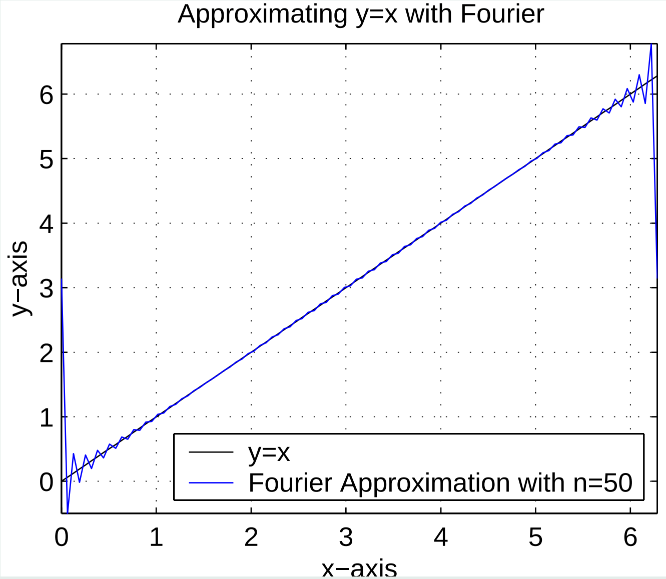

As clearly seen from the graphs, the more terms we use for basis for the space we project onto, the more the similarity between f and its projection. In other words, the more terms we include in the Fourier approximation, the better the approximation becomes.

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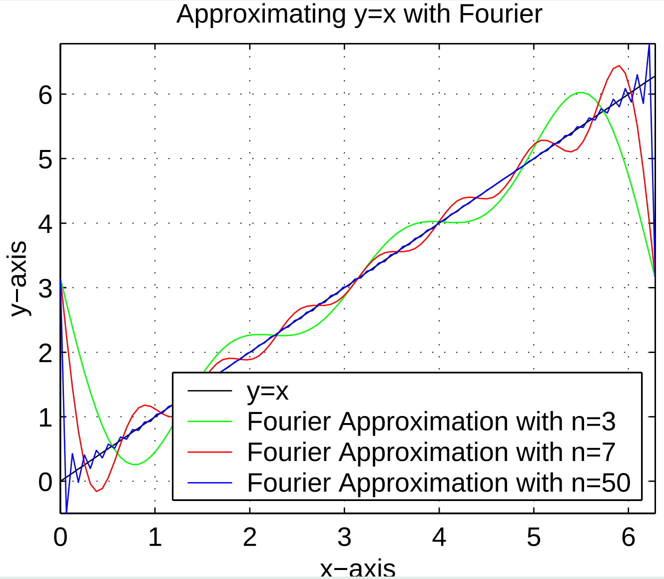

Figure 5: Convergence

Conclusion

We have shown that the vector space containing all continuous functions is an inner product space. The set of continuous functions, $\{1, \sin x, \cos x, \sin 2x, \cos 2x, \cdots, \sin nx, \cos nx, \cdots\}$, is mutually orthogonal on the vector space $C[0, 2\pi]$. We can create an orthonormal set by dividing each function by its magnitude. Then, by projecting a function on the space spanned by the set, we get the Fourier Approximation of that function. By graphical example, it has been shown that an increased number of terms in the projection decreases the error in the approximation.

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References

[1] Arnold, David. *Fall 2001, Lectures and insight by the master.*

[2] Polking, Boggess, and Arnold.*Differential Equations, 2001*

[3] Strang, Gilbert. *Introduction to Linear Algebra. 1998*

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