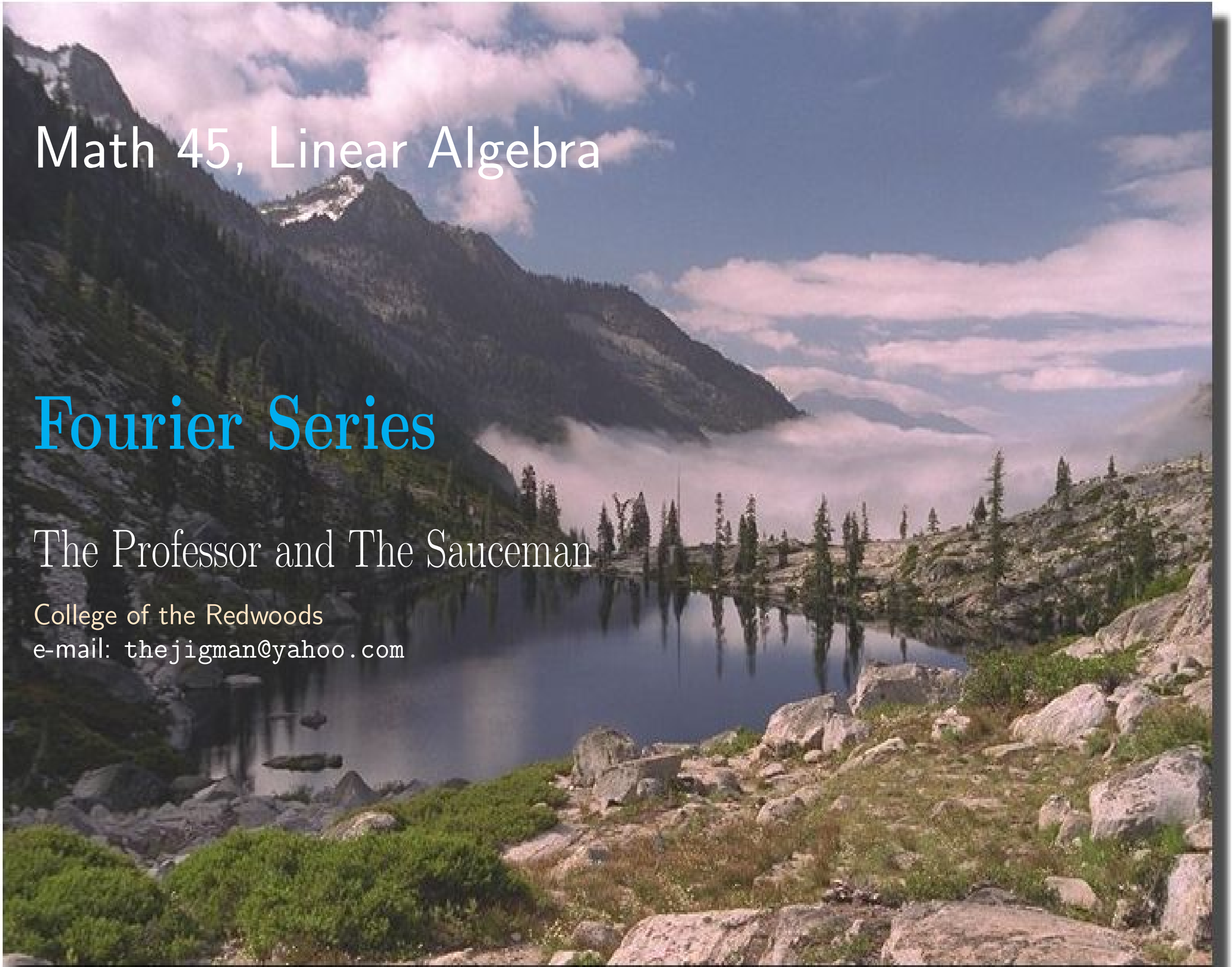

Math 45, Linear Algebra

Fourier Series

The Professor and The Sauceman

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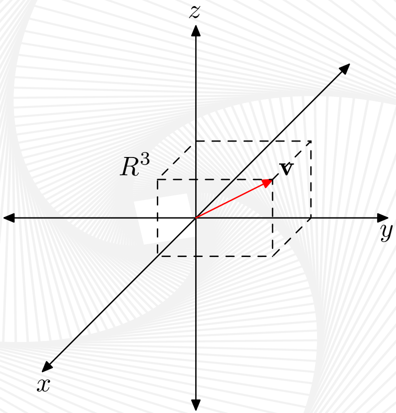
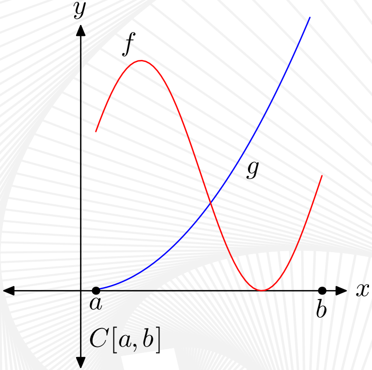
- **Create an orthonormal set by dividing each element by its magnitude.**
- **To mathematically derive the Fourier Series as a projection.**
- **An Example: Project a function onto the space spanned by our orthonormal set.**

The Vector Spaces

- A 3 dimensional vector lives in the vector space $\mathbb{R}^3$. Usually this is what we deal with in linear algebra, that or $\mathbb{R}^4$, $\mathbb{R}^n$, etc.



- All continuous functions also live in a vector space. On the interval from a to b, that space is notated as C[a,b].



- It would be easy to show that the ten properties of a vector space are satisfied by this space.
- For any vectors f and g in C[a,b], the inner product is notated as $< f, g >$ and is defined to be $\int_a^b f(x)g(x)dx$.

For C[a,b] to to be an inner product space, in addition to satisfying the ten properties of a vector space, $< f, g >$ must satisfy the following conditions of inner products:

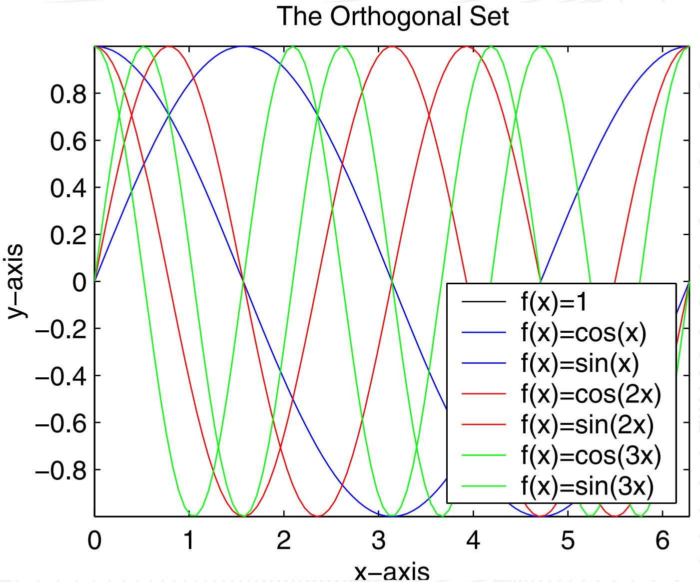
1. $< f, f > \geq 0$
2. $< f, f > = 0$ if and only if $f = 0$
3. $< f, g > = < g, f >$
4. $< \alpha f, g > = < f, \alpha g > = \alpha < f, g >$
5. $< f, g + h > = < f, g > + < f, h >$

By using the definition of an inner product, $< f, g > = \int_a^b f(x)g(x)dx$, each of these can be easily shown.

“Orthogonal is Good”

$$\{1,\ \cos x,\ \sin x,\ \cos 2x,\ \sin 2x,\ \cos 3x,\ \sin 3x,\ \cdots\}$$

- For this set to be orthogonal in C[0,2π], every single term must be orthogonal to every other one in that space.
- They certainly don’t look orthogonal, or at least not orthogonal as we are used to.



- However, if the inner product of any two of the functions is zero, the set is indeed orthogonal.

By evaluating the following three inner products, we can prove that it is an orthogonal set.

1. $< \sin nx, \sin mx >= 0, \quad$ for $\quad n \neq m$
2. $< \sin nx, \cos mx >= 0$
3. $< \cos nx, \cos mx >= 0, \quad$ for $\quad n \neq m$

- Note that by choosing the appropriate values for n and m, the inner product of any two terms in our set can be given by inner product 1, 2, or 3.

Working out these inner products is quite involved, involving integrals with nasty trigonometric identities. Here is the middle one worked out:

$$
\begin{aligned}
&< \sin nx, \cos mx >= \int_0^{2\pi} \sin nx \cos mx \, dx \\
&= \int_0^{2\pi} \frac{1}{2}[\sin(nx - mx) + \sin(nx + mx)]dx \\
&= \frac{1}{2}\int_0^{2\pi} \sin((n-m)x)dx + \frac{1}{2}\int_0^{2\pi} \sin((n+m)x)dx \\
&= \frac{1}{2}\left[-\frac{1}{n-m}\cos((n-m)x)dx\right]_0^{2\pi} \\
&\qquad + \frac{1}{2}\left[-\frac{1}{n+m}\cos((n+m)x)dx\right]_0^{2\pi} \\
&= -\frac{1}{2}\left(\frac{1}{n-m}\right)[\cos((n-m)2\pi) - \cos((n-m)0)] \\
&\qquad - \frac{1}{2}\left(\frac{1}{n+m}\right)[cos((n+m)2\pi) - cos((n+m)0)]
\end{aligned}
$$

$$\begin{aligned} < \sin nx, \cos mx > &= -\frac{1}{2}\left(\frac{1}{n-m}\right)[1-1] - \frac{1}{2}\left(\frac{1}{n+m}\right)[1-1] \\ &= 0. \end{aligned}$$

It turns out that no matter what the values of n and m, the inner product is zero, so the vectors are orthogonal. The other two inner products are evaluated similarly.

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- First calculate the squared magnitude of each term by taking the inner product of each vector with itself.
- The magnitudes squared of all sine vectors can be calculated in one inner product,

$$\begin{aligned} \|\sin nx\|^2 &= < \sin nx, \sin nx > \\ &= \int_0^{2\pi} (\sin nx)^2 dx \\ &= \pi \end{aligned}$$

- The square of the magnitude of the cosine functions can also be calculated with a single inner product, giving the same result as that of the sines:

$$
\begin{aligned}
\|\cos nx\|^2 &= < \cos nx, \cos nx > \\
&= \int_0^{2\pi} (\cos nx)^2 dx \\
&= \pi
\end{aligned}
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- The square of the magnitude of the vector 1, which is the first vector in the set, is different than the rest:

$$\begin{aligned} \|1\|^2 &=< 1, 1 > \\ &= \int_0^{2\pi} 1^2 dx \\ &= 2\pi \end{aligned}$$

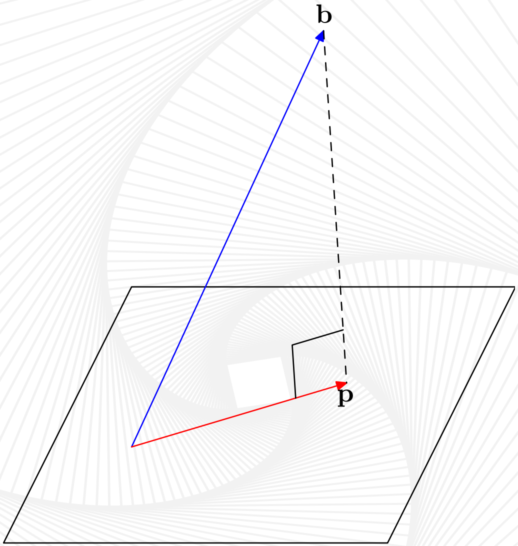
- Take the square roots of these values to get the magnitudes:
  - Magnitude of sine terms: $\sqrt{\pi}$
  - Magnitude of cosine terms: $\sqrt{\pi}$
  - Magnitude of 1: $\sqrt{2\pi}$
- Now divide everything in $\{1,\ \cos x,\ \sin x,\ \cos 2x,\ \sin 2x,\ \cos 3x,\ \cdots\}$ by its magnitude, and an orthonormal set results:

$$\left\{\frac{1}{\sqrt{2\pi}},\ \frac{1}{\sqrt{\pi}}\cos x,\ \frac{1}{\sqrt{\pi}}\sin x,\ \frac{1}{\sqrt{\pi}}\cos 2x,\ \frac{1}{\sqrt{\pi}}\sin 2x,\ \cdots\right\}.$$

Projecting a Function Onto the Space Spanned by the Orthonormal set

- When a vector lives outside a vector space, what vector within the space is closest to the outside vector? The answer to this is the projection of that vector onto the space.
- Using the vectors from our orthonormal set as the basis, we can form a space to project other functions onto.

- This is a little abstract, so first consider what it would mean to project a 3 dimensional vector onto a space.



- The projection of a function onto a space spanned by functions is the same idea, but graphically, it's quite different. First we'll show mathematically how to do it, then we'll give a graphical example.

The Math

First form a matrix Q with orthonormal columns. Using the idea of least squares approximation, we will find that the projection of $\mathbf{b}$ onto the space spanned by Q gives the vector in the space closest to $\mathbf{b}$. The formula for the projection onto the column space of Q is

$$\mathbf{p} = Q(Q^T Q)^{-1} Q^T \mathbf{b}.$$

Since Q is an orthogonal matrix, $(Q^T Q) = I$. Thus,

$$\begin{aligned}\mathbf{p} &= Q I^{-1} Q^T \mathbf{b} \\ &= Q Q^T \mathbf{b}.\end{aligned}$$

Continuing with our projection.

$$\mathbf{p} = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 & \cdots & \mathbf{q}_n \end{bmatrix} \begin{bmatrix} \mathbf{q}_1^T \\ \mathbf{q}_2^T \\ \mathbf{q}_3^T \\ \vdots \\ \mathbf{q}_n^T \end{bmatrix} \mathbf{b}.$$

$$\begin{aligned}\mathbf{p} &= \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 & \cdots & \mathbf{q}_{2n} \end{bmatrix} \begin{bmatrix} \mathbf{q}_1^T \\ \mathbf{q}_2^T \\ \mathbf{q}_3^T \\ \vdots \\ \mathbf{q}_n^T \end{bmatrix} \mathbf{b}. \\ &= (\mathbf{q}_1\mathbf{q}_1^T + \mathbf{q}_2\mathbf{q}_2^T + \mathbf{q}_3\mathbf{q}_3^T + \cdots + \mathbf{q}_n\mathbf{q}_n^T)\mathbf{b}\end{aligned}$$

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&= \mathbf{q}_1\mathbf{q}_1^T\mathbf{b} + \mathbf{q}_n\mathbf{q}_n^T\mathbf{b} + \mathbf{q}_3\mathbf{q}_3^T\mathbf{b} + \cdots + \mathbf{q}_n\mathbf{q}_n^T\mathbf{b}
\end{aligned}$$

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&=< \mathbf{b}, \mathbf{q}_1 > \mathbf{q}_1 + < \mathbf{b}, \mathbf{q}_2 > \mathbf{q}_2 + < \mathbf{b}, \mathbf{q}_3 > \mathbf{q}_3 + \\
&\qquad\qquad \cdots + < \mathbf{b}, \mathbf{q}_n > \mathbf{q}_n
\end{aligned}
$$

Where $< \mathbf{b}, \mathbf{q}_k >= \mathbf{q}_k^T\mathbf{b}$.

Similarly, because $V = C[0, 2\pi]$ is an innerproduct space, the projection of f onto the space spanned by our orthonormal set is

$$\begin{aligned} p =&< f, \frac{1}{\sqrt{2\pi}} > \frac{1}{\sqrt{2\pi}} \\ &+ < f, \frac{1}{\sqrt{\pi}} \cos x > \frac{1}{\sqrt{\pi}} \cos x + < f, \frac{1}{\sqrt{\pi}} \sin x > \frac{1}{\sqrt{\pi}} \sin x \\ &+ \cdots \\ &+ < f, \frac{1}{\sqrt{\pi}} \cos nx > \frac{1}{\sqrt{\pi}} \cos nx + < f, \frac{1}{\sqrt{\pi}} \sin nx > \frac{1}{\sqrt{\pi}} \sin nx. \end{aligned}$$

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&+ \cdots \\
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\end{aligned}
$$

Since all of the inner products produce constants, we label each innerproduct with an a_k and a b_k.

$$p = a_0 + a_1 \cos x + b_1 \sin x + \cdots + a_n \cos nx + b_n \sin nx$$

- The constants $a_0,\ a_1,\ b_1,\ \cdots,\ a_n,\ b_n$ are the *Fourier coefficients* of f, and are given by

$$a_0 =< f, \frac{1}{\sqrt{2\pi}} > \frac{1}{\sqrt{2\pi}} = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$$

$$a_k =< f, \frac{1}{\sqrt{\pi}}\cos kx > \frac{1}{\sqrt{\pi}} = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx dx$$

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- This projection of f onto the space spanned by

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos x, \frac{1}{\sqrt{\pi}}\sin x, \cdots, \frac{1}{\sqrt{\pi}}\cos nx, \frac{1}{\sqrt{\pi}}\sin nx\right\}$$

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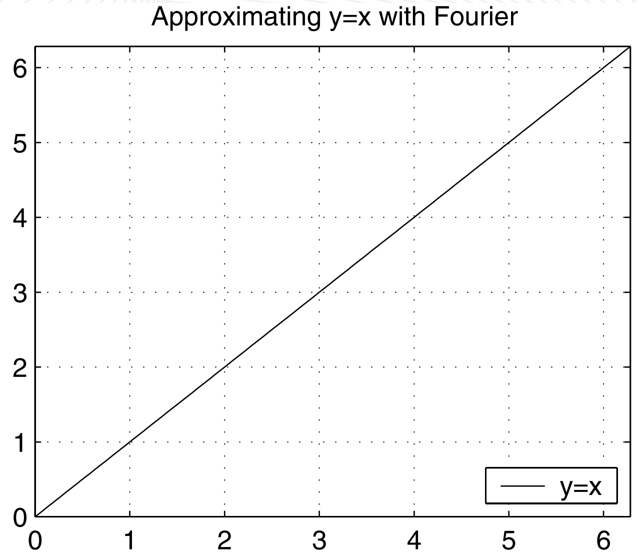
A Graphical Example

Now that we've shown mathematically how to derive the Fourier Series, we'll take the function $f(x) = x$, and show its Fourier Series graphically.



- To begin with, we need to calculate the fourier coefficients for $f(x) = x$,

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} x dx$$
$$a_k = \frac{1}{\pi}\int_0^{2\pi} x \cos kx dx$$
$$b_k = \frac{1}{\pi}\int_0^{2\pi} x \sin kx dx$$

- Evaluating these integrals gives

$$a_0 = \pi$$
$$a_k = 0$$
$$b_k = -\frac{2}{k}.$$

- Because a_k, the Fourier Coefficient associated with cosine, is 0 all of the cosine terms are removed.

- Then the projection of $f(x) = x$ onto the space spanned by our complete set becomes

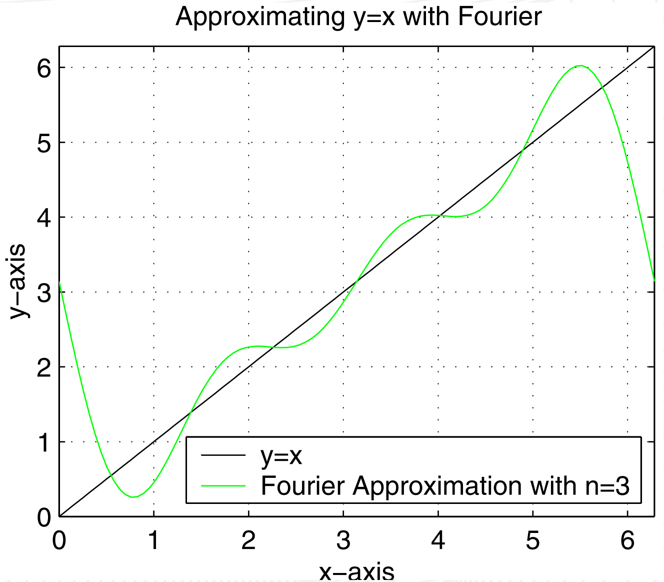
$$p = \pi - 2\sin x - \sin 2x - \frac{2}{3}\sin 3x - \frac{1}{2}\sin 4x - \frac{2}{5}\sin 5x - \cdots$$

- If we only use the first four vectors in our set as the basis for the space we project onto,

$$\mathbb{B} = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos x, \frac{1}{\sqrt{\pi}}\sin x, \frac{1}{\sqrt{\pi}}\cos 2x \right\},$$

we get the Fourier series with four terms, up through $n = 3$,

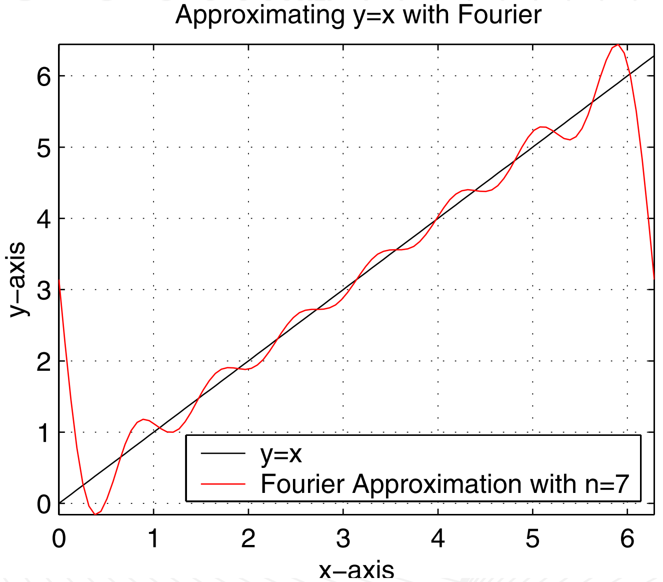
$$p = \pi - 2\sin x - \sin 2x - \frac{2}{3}\sin 3x.$$



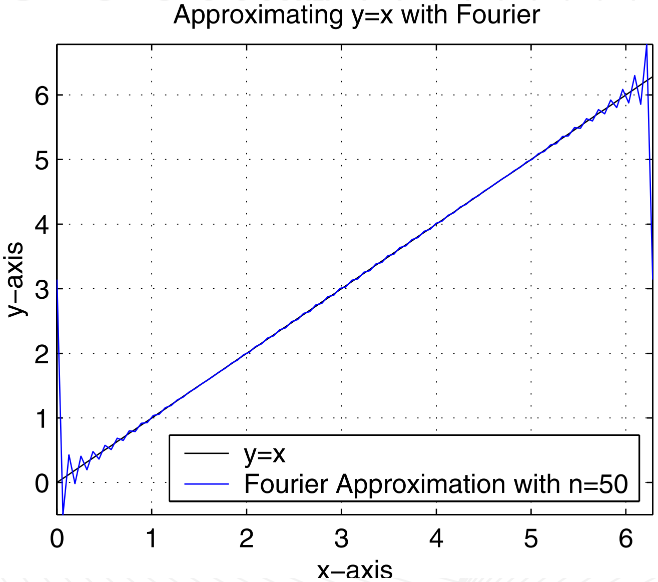
- Note that we are only dealing with the vector space $C[0, 2\pi]$.

Outside of that space has nothing to do with the projection.

- Now if we use the first 8 terms, up through $n = 7$, we get a function that looks even more like f.



- If the space we project onto has all the terms up through $n = 50$ as the basis, the projection looks almost identical to f.



- As clearly seen from the graphs, the more terms we use for basis for the space we project onto, the more the similarity between f and its projection. In other words, the more terms we include in the fourier approximation, the better the approximation becomes.

Applications of the Fourier Series

- There are a wide range of application of Fourier Analysis within pure mathematics. Such subjects are:
  - Finding the sum of a series
  - Isoperimetric problems
  - Differential Equations
  - Statistics
  - The prime number theorem

- There are also applications within the field of physics. Some examples are:
  - Quantum theory
  - Lasers
  - Electron Scattering
  - Diffraction
  - Telescopes
  - Impedance

- Some applications in Chemistry include:
  - Mass spectrometry
  - NMR spectroscopy
  - Infra-red spectroscopy
  - Visible light spectroscopy
  - Crystallography

- There even application in the life sciences. Some of which are:
  - Vision
  - Hearing
  - Speech Analysis
  - Morphogenesis
  - Medical

- The applications of Fourier Analysis extends to many areas of the sciences. Some miscellaneous applications are:
  - Water Waves
  - Turbulence in fluids
  - Meteorology
  - Glacier beds and “roughness”
  - Seismology
  - Vibration analysis
  - Economics
- The list goes on and on.

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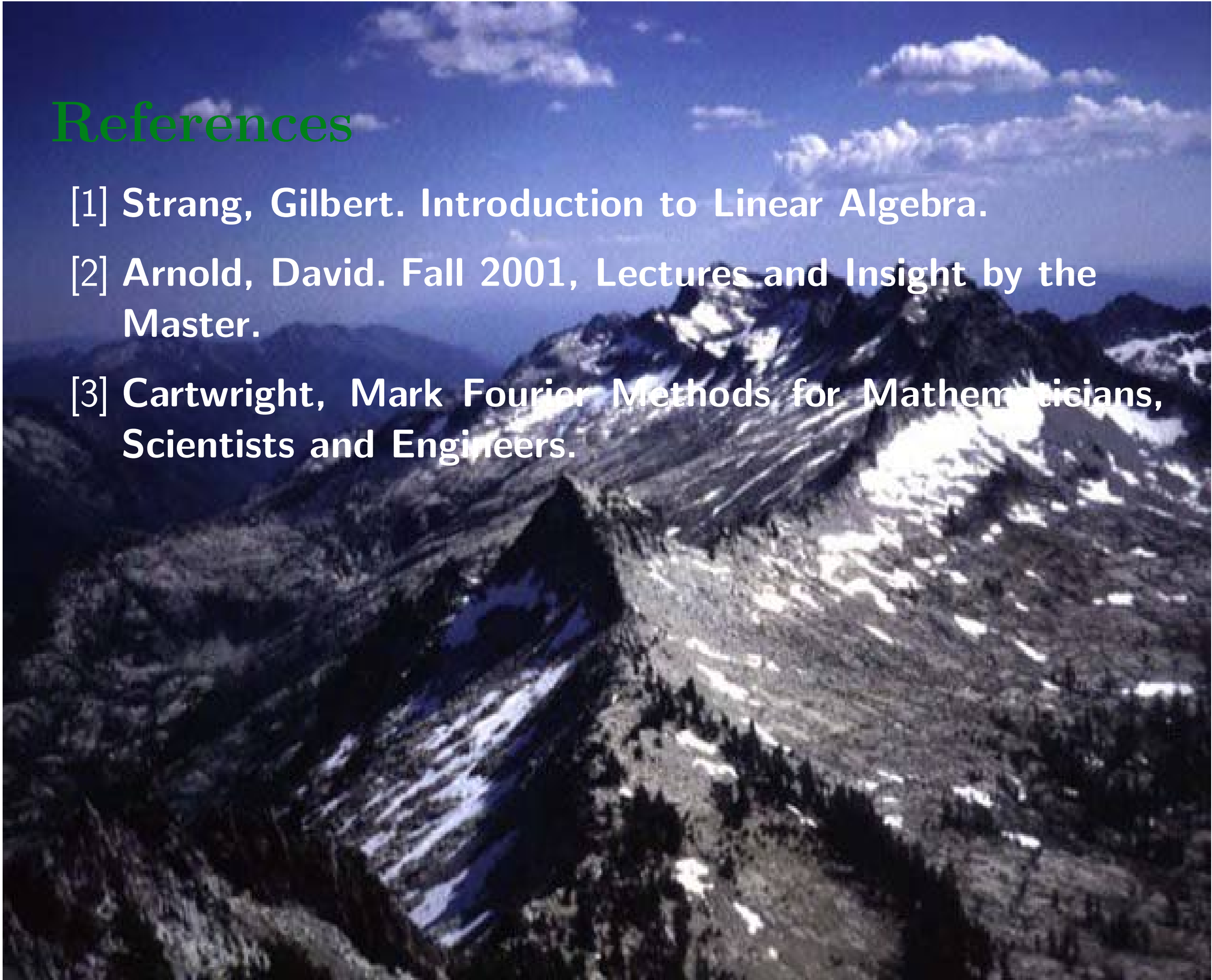
- An orthonormal set is created by dividing each element by its magnitude.
- The mathematical derivation of the Fourier Series is a projection.
- The graphical example shows that an increased number of terms in the projection decreases the error in the approximation.



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[3] **Cartwright, Mark Fourier Methods for Mathematicians, Scientists and Engineers.**