The Solving of Lights Out

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December 17, 2001

Abstract

The purpose of this paper is to discuss the solving of the game *Lights Out!* using Linear Algebra.

1. Introduction

Lights Out! is a puzzle-type game manufactured by *Tiger Electronics*. It consists of a 5×5 board of buttons which can be lit up. When a button is pushed, it's state changes as well as the state of all the buttons adjacent to it. If a button was lighted up, when pushed, it turns off, if a button starts with it's light off, pushing the button turns it on. For example, if the beginning state of the game board is all the lights being out, then pushing the blue button indicated below causes it and all the yellow buttons to light up.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Pushing the next button indicated in blue causes the following changes.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

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The final state of the board is:

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Notice that the effect of the blue buttons being pressed is not changed when the buttons are pressed in reverse order. The order of presses does not matter! The goal of the game is, starting with any given state of the game board, any random combination of the lights being on, to turn all the lights off, or should we say, *out*.

2. Playing with Linear Algebra

For facility of explaining the game in terms of vectors and matrices, let the buttons on the board be numbered in the following way:

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

The game board can be depicted as a 25×1 vector with the first term representing button 1, the second term representing button 2, and so on. For the simplest way to represent the game, the vector has only 1's and 0's in it. 1 represents a button with its light on and 0 represent a button with its light off. Suppose the initial state of the board is all lights are out. This is a vector of all 0's. If button 5 is pressed, then it lights up along with adjacent buttons 4 and 10. Here's the vector $\vec{a_5}$ that represents what happens when button 5 is pressed.

$$\vec{a}_5 = \begin{array}{ccccc} (0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0) \end{array}$$

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This is a vector arranged in a 5×5 fashion for the sake of space and clarity as to what is happening. This will be done throughout the paper. Similarly,

$$\vec{a}_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Each button i's action can be represented by a vector $\vec{a}_i$ that shows what other buttons are changed when button i is pressed. The effect of pushing a combination of buttons can be found by adding the "effect" vectors, the $\vec{a}_i$'s for the buttons pushed.

2.1. Important Note:

Before going any farther, some special addition operators must be defined for the set of vectors and matrices that depict this game. The number zero represents a light being out and the number one represents a light being on. No other numbers are needed in this set. We want all operations on any of the vectors that describe this game to produce vectors with only 0's and 1's in them. No other numbers would make any sense. Thus, this is a set in $(\mathbb{Z}_2)^{25}$ where $\mathbb{Z}_2$ is the subspace containing only 1 and 0. So here is addition in this subspace:

$$\begin{aligned} 0 + 1 &= 1 \\ 1 + 0 &= 1 \\ 0 + 0 &= 0 \\ 1 + 1 &= 0 \end{aligned}$$

These totally make sense in the context of the game. Changing the state of an unlit button results in a lit button, as well as not changing the state of a lit button. Not changing the state of an unlit button results in an unlit button. And finally, why $1 + 1 = 0$, changing the state of a lit button results in an unlit button. Also, pushing a button twice leaves the game board in the exact same state as before the button was pushed. So $2(1) = 0$ which follows anyway, from the definition of addition. Any even multiple of 1 in this subspace equals 0. Pushing a button any odd number of times gives the same effect as pushing the button once. Any odd multiple of 1 in this subspace equals 1. This means that no button must ever be pushed more than once for the game to be solved!

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2.2. The Matrix R

Adding a combination of the 25×1 vectors $\vec{a_i}$ to get the effect of pushing that combination of buttons could give a really long equation. Here is another way to do the same thing. If all the vectors $\vec{a}$ are put into a matrix R, we have:

$$R = \begin{bmatrix} a_1 & a_2 & \cdots & a_{25} \end{bmatrix}$$

This 25×25 matrix encapsules all the rules of the game. The vectors $\vec{a}$ that make up the columns of R form the following
pattern:

$$R = \begin{pmatrix} A & I & O & O & O \\ I & A & I & O & O \\ O & I & A & I & O \\ O & O & I & A & I \\ O & O & O & I & A \end{pmatrix}$$

with

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

and I and O being the 5×5 identity and zero matrices. Let a vector $\vec{p}$ made up of 0's and 1's be called the press vector (the 1's represent the buttons that are pressed). Multiplying this by the matrix R gives the vector $R\vec{p}$, the *effect* of $\vec{p}$. Let the vector $\vec{s}$ represent the initial state of the game board. Then $R\vec{p} + \vec{s}$ is the state of the board after pressing the buttons chosen in $\vec{p}$. So the goal of the game is, given an initial state $\vec{s}$ to find $\vec{p}$ so that

$$R\vec{p} + \vec{s} = \vec{0}$$

2.3. Finding the solution to $R\vec{p} + \vec{s} = \vec{0}$

Because of the special operations of addition for the set, for any vector $\vec{s}$ in the set, $\vec{s} + \vec{s} = \vec{0}$. Thus,

$$R\vec{p} = \vec{s}$$

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So solving for a given state $\vec{s}$ is the same as reaching that state from a totally unlit game board. Some of the properties of matrix R can give some insights in to the solution of this problem. The last two properties were found from performing Gausssian elimination on the matrix.

- R is a symmetric, 25×25 matrix.
- $C(R)$, the range of R has dimension 23. The last 2 columns of R are the "free" columns. So all solutions can be written as linear combinations of the first 23 rows.
- Then the nullspace of R, corresponding to the kernel of R, has dimension 2. A basis for this nullspace can be found through the last 2 columns of R in it's reduced-row echelon form.

2.4. The Question

Is there a solution for every possible initial state $\vec{s}$ in $R\vec{p} = \vec{s}$? For there to be a solution, $\vec{s}$ must be in column space of R. Thus there are some initial states $\vec{s}$ for which there is no solution to $R\vec{p} = \vec{s}$. (The column space of R is not all of $(\mathbb{Z}_2)^{25}$.) A criterion for the existence of a solution can be found through the nullspace of R. A basis for the nullspace of R is:

$$B = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

For an initial state $\vec{s}$ to be solvable, it must be orthogonal to these basis vectors for the nullspace of matrix R! Also, when it is solvable, there are 4 different solutions possible.

2.5. Four Different Solutions

Putting R and (a solvable!) $\vec{s}$ into an augmented matrix $[R\vec{s}]$ and then performing Gaussian elimination to reach the reduced row echelon form, gives the particular solution, the 26 column of the reduced matrix. The complete solution is the particular solution plus any linear combination of the nullspace vectors. There are only three non-trivial combinations of the basis vectors for the nullspace in this set in $(\mathbb{Z}_2)^{25} = \{0, 1\}^{25}$. Calling the two basis vectors for $N(R)$ $\vec{b_1}$ and $\vec{b_2}$, the 4 solutions to

$$R\vec{p} + \vec{s} = \vec{0}$$

are:

$$\vec{P}_{particular}$$
$$\vec{P}_{particular} + \vec{b_1}$$
$$\vec{P}_{particular} + \vec{b_2}$$
$$\vec{P}_{particular} + \vec{b_1} + \vec{b_2}$$

3. One Example of Solving Lights Out

Starting State:

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Solve $R\vec{p} = \vec{s}$ with

$$s = \begin{matrix} (1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1) \end{matrix}$$

3.1. The Solution

With the augmented matrix $[R\vec{s}]$ we do Gauss-Jordan elimination to reach reduced-row echelon. The last column is the particular solution, which for this case is:

$$p = \begin{matrix} (0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1) \end{matrix}$$

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$R\vec{p} = \vec{s}$ and also $R\vec{p} = \vec{b_1} = \vec{s}$,$R\vec{p} + \vec{b_2} = \vec{s}$, and $R\vec{p} + \vec{b_1} + \vec{b_2} = \vec{s}$ $\vec{p}$ shows that if buttons 2, 10, 17, 22, and 25 are pressed, all the lights will go out! The other three solutions are:

$$\vec{p} + \vec{b_1} = \begin{array}{ccccc} (0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1) \end{array}$$

pushing buttons 3, 4, 6, 8, 11, 12, 14, 15, 16, 17, 18, 20, 23, 24,and 25

$$\vec{p} + \vec{b_2} = \begin{array}{ccccc} (1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0) \end{array}$$

pushing buttons 1, 2, 3, 5, 6, 8, 16, 17, 18, 20, 21, 22, and 23

$$\vec{p} + \vec{b_1} + \vec{[}b_2 = \begin{array}{ccccc} (1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0) \end{array}$$

pushing buttons 1, 4, 5, 10, 11, 12, 14, 15, 17, 21, and 24.

4. A Discovery

It would be nice to find a simpler way to solve this problem without having to mess around with these rather large vectors and matrices. Let's write the problem as follows:

$$R \begin{pmatrix} \vec{p_1} \\ \vec{p_2} \\ \vec{p_3} \\ \vec{p_4} \\ \vec{p_5} \end{pmatrix} = \begin{pmatrix} \vec{s_1} \\ \vec{s_2} \\ \vec{s_3} \\ \vec{s_4} \\ \vec{s_5} \end{pmatrix}$$

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where $\vec{p}$ and $\vec{s}$ have been broken up into 5×1 subvectors. $R\vec{p} = \vec{s}$ can be manipulated to the equivalent equation:

$$J\vec{p} = (R + J)\vec{p} + \vec{s}$$

for any matrix J.

4.1. Linear Dependence

Using

$$J = \begin{pmatrix} O & I & O & O & O \\ O & O & I & O & O \\ O & O & O & I & O \\ O & O & O & O & I \\ O & O & O & O & O \end{pmatrix}$$

The equation becomes:

$$\begin{pmatrix} \vec{p_2} \\ \vec{p_3} \\ \vec{p_4} \\ \vec{p_5} \\ \vec{0} \end{pmatrix} = \begin{pmatrix} A & O & O & O & O \\ I & A & O & O & O \\ O & I & A & O & O \\ O & O & I & A & O \\ O & O & O & I & A \end{pmatrix} \begin{pmatrix} \vec{p_1} \\ \vec{p_2} \\ \vec{p_3} \\ \vec{p_4} \\ \vec{p_5} \end{pmatrix} + \begin{pmatrix} \vec{s_1} \\ \vec{s_2} \\ \vec{s_3} \\ \vec{s_4} \\ \vec{s_5} \end{pmatrix}$$

Each $\vec{p_i}$ is a combination of the subvectors $\vec{p_k}$ and $\vec{s_k}$ with $k < i$.

$$\vec{p_2} = A\vec{p_1} + \vec{s_1}$$

$$\vec{p_3} = \vec{p_1} + A\vec{p_2} + \vec{s_2}$$
$$\vec{p_4} = \vec{p_2} + A\vec{p_3} + \vec{s_3}$$
$$\vec{p_5} = \vec{p_3} + A\vec{p_4} + \vec{s_4}$$

Each $\vec{p_2} \ldots \vec{p_5}$ can therefore be expressed as a combination of only $\vec{p_1}$ and $\vec{s}$.

$$\begin{pmatrix} \vec{p_2} \\ \vec{p_3} \\ \vec{p_4} \\ \vec{p_5} \\ \vec{0} \end{pmatrix} = \begin{pmatrix} B_1 & B_0 & O & O & O & O \\ B_2 & B_1 & B_0 & O & O & O \\ B_3 & B_2 & B_1 & B_0 & O & O \\ B_4 & B_3 & B_2 & B_1 & B_0 & O \\ B_5 & B_4 & B_3 & B_2 & B_1 & B_0 \end{pmatrix} \begin{pmatrix} \vec{p_1} \\ \vec{s_1} \\ \vec{s_2} \\ \vec{s_3} \\ \vec{s_4} \\ \vec{s_5} \end{pmatrix}$$

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with $B_0 = I$,$B_1 = A$, and $B_{n+2} = AB_{n+1} + B_n$. Taking the last row of this matrix,

$$B_5\vec{p_1} = B_4\vec{s_1} + B_3\vec{s_2} + B_2\vec{s_3} + B_1\vec{s_4} + B_0\vec{s_5}$$

a new way to solve the equation using vectors and matrices in $(\mathbb{Z}_2)^5$ instead of $(\mathbb{Z}_2)^{25}$ is seen.

4.2. Solving the new equation

$$B_5\vec{p_1} = B_4\vec{s_1} + B_3\vec{s_2} + B_2\vec{s_3} + B_1\vec{s_4} + B_0\vec{s_5}$$

- Calculate the right-hand side of the equation, which depends only on $\vec{s}$.
- Find $\vec{p_1}$ so that $B_5\vec{p_1}$ equals the right-hand side.
- Lastly, calculate $\vec{p_2}$...$\vec{p_5}$ directly from $\vec{p_1}$ and $\vec{s}$ with the four first rows of the matrix.

5. Conclusion

Using Linear Algebra to solve this game is just one of the many ways Linear Algebra finds a place in everyday life. This application is not very complicated or technical, utilizing mainly just the very basics of Linear algebra, with just a slightly different subspace than usual. But it shows how versatile Linear Algebra is. And Linear Algebra can be fun!

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