Math 45 – Linear Algebra Project

Lights Out!

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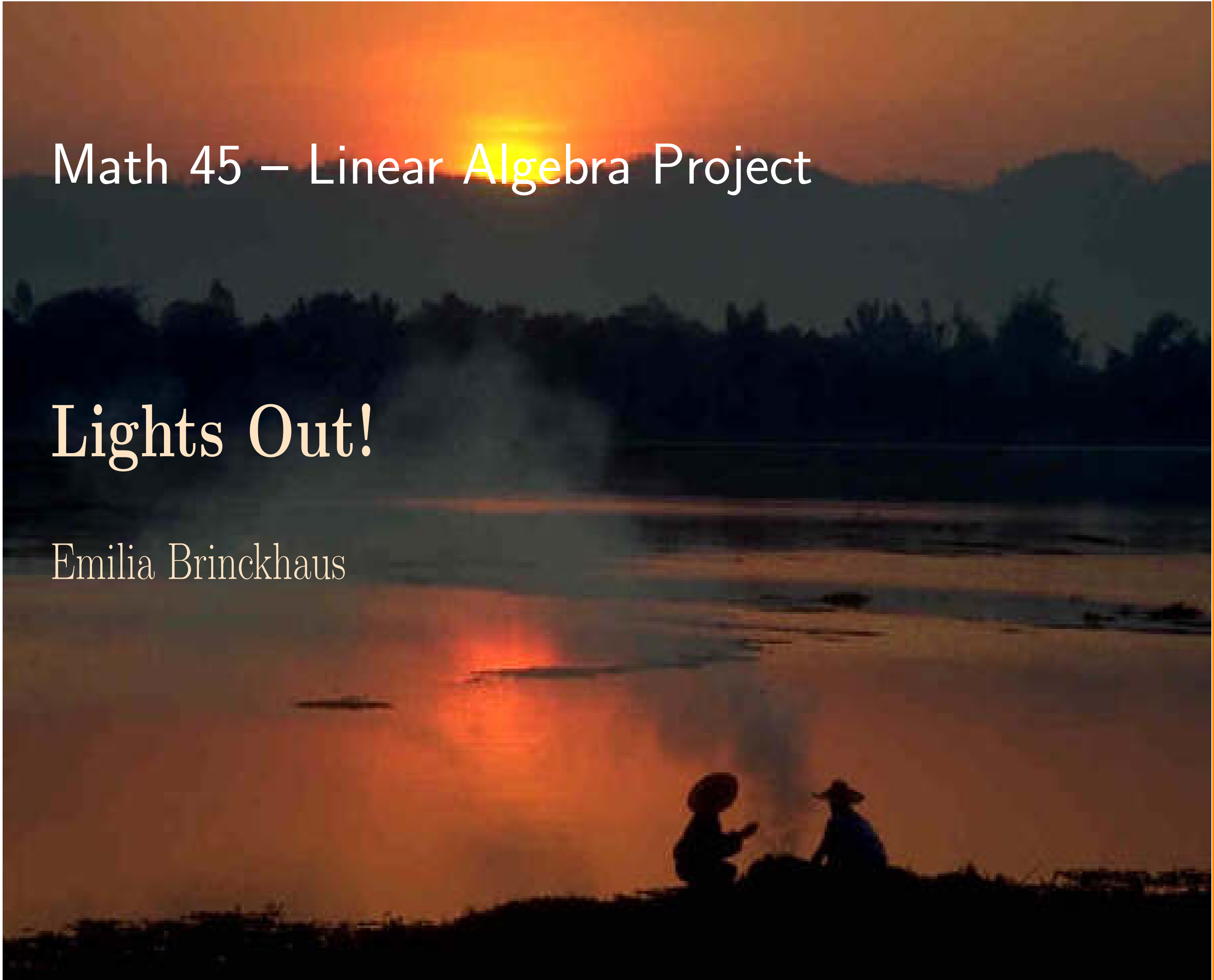

The Game

What it is:

Lights Out is an electronic puzzle-type game that is made up of a 5×5 board of buttons that can light up.

How it works:

When one of the buttons is pushed, its state changes, as well as the state of all the buttons adjacent to it.

The Goal:

The main point of the game is, starting with any random combination of the lights on, to turn all the lights off. A side point is to do this with the fewest number of buttons pressed as possible.

Pushing Buttons

Starting with a completely unlit game board, here is the effect of pressing the following buttons: (Order does not matter!)

Button 2

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Button 7

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Using Linear Algebra to play Lights Out

For facility of explaining the game in terms of vectors and matrices, let the buttons on the board be numbered in the following way:

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

The game board can be depicted as a 25×1 vector with only 1's and 0's in it. 1 represents a button with its light on and 0 represent a button with its light off. Suppose the initial state of the board is all lights are out. This is a vector of all 0's. If button 5 is pressed, then it lights up along with adjacent buttons 4 and 10.

Here's the vector $\mathbf{a}_5$ that represents what happens when button 5 is pressed.

$$\mathbf{a}_5 = \begin{matrix} (0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0) \end{matrix}$$

This is a vector arranged in a 5×5 fashion for the sake of space and clarity as to what is happening. Similarly,

$$\mathbf{a}_{12} = \begin{matrix} (0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0) \end{matrix}$$

Each button i's action can be represented by a vector $\mathbf{a}_i$ that shows what other buttons are changed when button i is pressed.

The Matrix R

If all the vectors $\mathbf{a}$ are put into a matrix R, we have:

$$R = \begin{bmatrix} a_1 & a_2 & \cdots & a_{25} \end{bmatrix}$$

This 25×25 matrix encapsules all the rules of the game.

Vector p

Let a vector $\mathbf{p}$ made up of 0's and 1's be called the press vector (the 1's represent the buttons that are pressed). Multiplying this by the matrix R gives the vector $R\mathbf{p}$, the *effect* of $\mathbf{p}$.

Vector s

Let the vector $\mathbf{s}$ represent the initial state of the game board. Then $R\mathbf{p} + \mathbf{s}$ is the state of the board after pressing the buttons chosen in $\mathbf{p}$. So the goal of the game is, given an initial state $\mathbf{s}$ to find $\mathbf{p}$ so that

$$R\mathbf{p} + \mathbf{s} = \mathbf{0}$$

Important Side-note

We have to define some special addition operations for this set of 0's and 1's which is in $(\mathbb{Z}_2)^{25}$.

Addition modulo 2

$$0 + 1 = 1$$
$$1 + 0 = 1$$
$$0 + 0 = 0$$

as usual. But also

$$1 + 1 = 0.$$

$(\mathbb{Z}_2)^{25} = \{0, 1\}^{25}$, with addition modulo 2 for this set. These definitions must be made for the rules of the game to apply.

Finding the solution to:

$$R\mathbf{p} + \mathbf{s} = \mathbf{0}$$

Because of the special operations of addition for the set, for any vector $\mathbf{s}$ in the set, $\mathbf{s} + \mathbf{s} = \mathbf{0}$. Thus,

$$R\mathbf{p} = \mathbf{s}$$

So solving for a given state $\mathbf{s}$ is the same as reaching that state from a totally unlit game board. The properties of matrix R can give some insights in to the solution of this problem.

The matrix R

The vectors $\mathbf{a}$ that make up the columns of R form the following pattern:

$$R = \begin{pmatrix} A & I & O & O & O \\ I & A & I & O & O \\ O & I & A & I & O \\ O & O & I & A & I \\ O & O & O & I & A \end{pmatrix}$$

with

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

and I and O being the 5×5 identity and zero matrices.

Some properties of R

- R is a symmetric, 25×25 matrix.

- $C(R)$, the range of R has dimension 23. The last 2 columns of R are the "free" columns.

- Then the nullspace of R, corresponding to the kernel of R, has dimension 2. A basis for this nullspace can be found through the last 2 columns of R in it's reduced-row echelon form.

The Question

Is there a solution for every possible initial state $\mathbf{s}$ in $R\mathbf{p} = \mathbf{s}$? For there to be a solution, $\mathbf{s}$ must be in column space of R. Thus there are some initial states $\mathbf{s}$ for which there is no solution to $R\mathbf{p} = \mathbf{s}$. A criterion for the existence of a solution can be found through the nullspace of R. A basis for the nullspace of R is:

$$B = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

For an initial state $\mathbf{s}$ to be solvable, it must be orthogonal to these basis vectors for the nullspace of matrix R! Also, when it is solvable, there are 4 different solutions possible.

Four Different Solutions

Putting R and (a solvable!) $\mathbf{s}$ into an augmented matrix $[R\mathbf{s}]$ and then performing Gaussian elimination to reach the reduced row echelon form, gives the particular solution, the 26 column of the reduced matrix. The complete solution is the particular solution plus any linear combination of the nullspace vectors. There are only three non-trivial combinations of the basis vectors for the nullspace in this set in $(\mathbb{Z}_2)^{25} = \{0,1\}^{25}$. Calling the two basis vectors for $N(R)$ $\mathbf{b_1}$ and $\mathbf{b_2}$, the 4 solutions to

$$R\mathbf{p} + \mathbf{s} = \mathbf{0}$$

are:

$$\mathbf{P_{particular}}$$
$$\mathbf{P_{particular}} + \mathbf{b_1}$$
$$\mathbf{P_{particular}} + \mathbf{b_2}$$
$$\mathbf{P_{particular}} + \mathbf{b_1} + \mathbf{b_2}$$

One Example of Solving Lights Out

Starting State:

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Solve $R\mathbf{p} = \mathbf{s}$ with

$$s = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

The Solution

With the augmented matrix $[R\mathbf{s}]$ we do Gauss-Jordan elimination to reach reduced-row echelon. The last column is the particular solution, which for this case is:

$$p = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$R\mathbf{p} = \mathbf{s}$ and also $R\mathbf{p} = \mathbf{b_1} = \mathbf{s}, R\mathbf{p} + \mathbf{b_2} = \mathbf{s}$, and $R\mathbf{p} + \mathbf{b_1} + \mathbf{b_2} = \mathbf{s}$ $\mathbf{p}$ shows that if buttons 2,10,17,22, and 25 are pressed, all the lights will go out!

A Discovery

It would be nice to find a simpler way to solve this problem without having to mess around with these rather large vectors and matrices. Let's write the problem as follows:

$$R \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \\ \mathbf{p}_4 \\ \mathbf{p}_5 \end{pmatrix} = \begin{pmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \\ \mathbf{s}_3 \\ \mathbf{s}_4 \\ \mathbf{s}_5 \end{pmatrix}$$

where $\mathbf{p}$ and $\mathbf{s}$ have been broken up into 5×1 subvectors. $R\mathbf{p} = \mathbf{s}$ can be manipulated to the equivalent equation:

$$J\mathbf{p} = (R + J)\mathbf{p} + \mathbf{s}$$

for any matrix J.

Linear Dependence

Using

$$J = \begin{pmatrix} O & I & O & O & O \\ O & O & I & O & O \\ O & O & O & I & O \\ O & O & O & O & I \\ O & O & O & O & O \end{pmatrix}$$

The equation becomes:

$$\begin{pmatrix} \mathbf{p}_2 \\ \mathbf{p}_3 \\ \mathbf{p}_4 \\ \mathbf{p}_5 \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} A & O & O & O & O \\ I & A & O & O & O \\ O & I & A & O & O \\ O & O & I & A & O \\ O & O & O & I & A \end{pmatrix} \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \\ \mathbf{p}_4 \\ \mathbf{p}_5 \end{pmatrix} + \begin{pmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \\ \mathbf{s}_3 \\ \mathbf{s}_4 \\ \mathbf{s}_5 \end{pmatrix}$$

Each $\mathbf{p_i}$ is a combination of the subvectors $\mathbf{p_k}$ and $\mathbf{s_k}$ with $k < i$.

Each $\mathbf{p_2} \ldots \mathbf{p_5}$ can therefore be expressed as a combination of only $\mathbf{p_1}$ and $\mathbf{s}$.

$$\begin{pmatrix} \mathbf{p_2} \\ \mathbf{p_3} \\ \mathbf{p_4} \\ \mathbf{p_5} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} B_1 & B_0 & O & O & O & O \\ B_2 & B_1 & B_0 & O & O & O \\ B_3 & B_2 & B_1 & B_0 & O & O \\ B_4 & B_3 & B_2 & B_1 & B_0 & O \\ B_5 & B_4 & B_3 & B_2 & B_1 & B_0 \end{pmatrix} \begin{pmatrix} \mathbf{p_1} \\ \mathbf{s_1} \\ \mathbf{s_2} \\ \mathbf{s_3} \\ \mathbf{s_4} \\ \mathbf{s_5} \end{pmatrix}$$

with $B_0 = I$, $B_1 = A$, and $B_{n+2} = AB_{n+1} + B_n$. Taking the last row of this matrix,

$$B_5\mathbf{p_1} = B_4\mathbf{s_1} + B_3\mathbf{s_2} + B_2\mathbf{s_3} + B_1\mathbf{s_4} + B_0\mathbf{s_5}$$

a new way to solve the equation using vectors and matrices in $(\mathbb{Z}_2)^5$ instead of $(\mathbb{Z}_2)^{25}$ is seen.

Solving the new equation

$$B_5\mathbf{p_1} = B_4\mathbf{s_1} + B_3\mathbf{s_2} + B_2\mathbf{s_3} + B_1\mathbf{s_4} + B_0\mathbf{s_5}$$

- Calculate the right-hand side of the equation, which depends only on $\mathbf{s}$.
- Find $\mathbf{p_1}$ so that $B_5\mathbf{p_1}$ equals the right-hand side.
- Lastly, calculate $\mathbf{p_2}...\mathbf{p_5}$ directly from $\mathbf{p_1}$ and $\mathbf{s}$ with the four first rows of the matrix.

Conclusion

Solving this game shows one of the many ways Linear Algebra finds a place in everyday life. This application is not very complicated or technical, utilizing mainly just the very basics of Linear algebra. It is awesome to see though, how many diverse applications Linear Algebra has. Linear Algebra remains one of the leaders of all mathematics!

Acknowledgements

1. David Arnold, Knowledge of LaTeX and Linear Algebra.

2. Carsten Haese, paper in the sci.math newsgroup (1998).

3. Oscar Martin-Sanchez and Cristobal Pareja-Flores, Two Reflected Analyses of Lights out, *Mathematics Magazine* Vol.74, No.4, (October 2001).

4. Gilbert Strang *Introduction to Linear Algebra.*