The Leontief Open Production Model or Input-Output Analysis

Iris Jensen

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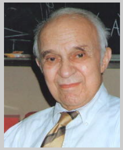
Abstract

Wassily Leontief won a Nobel Prize in Economics in 1973 for him explanation of the economy using his input-output model. There are two application of the Leontief model:a closed model and an open model. A closed model deals only with the income of each industry whereas the open model finds the amount of production needed to satisfy an increase in demand. Example one will familiarize you with the terminology and what different vectors represent. Using the same vector names as in the example we run through the linear algebra. Then the technology matrix from 1992 will be used to analyze the interdependencies among the sectors. The most useful application of input-output analysis for the economist or a common broker is the ability to be able to see how the change in demand for one industry effects the entire economy. I will focus on showing that the same total increase of demand will have different effects on the Gross Domestic Product, when the demand is added to different industries.

1. Introduction

Ever wonder how the government can predict a deeper recession when the airline industry shrinks or why certain antiquated industries are subsidized by the government? Input-output analysis can be applied to any size economy from a business district to the entire world. It is most often used for city planning and analysis of our national economy. The Leontief Open Production Model provides us with a powerful economic analysis tool in the form of input-output analysis. Input-Output analysis the modern name for the process of manipulating a Leontief open production model. Before you can use this tool you must understand the linear algebra behind it. Gaining this basic understanding is the purpose of the simple example.



2. A Simple Example

The technology matrix A will describe the relations a sector has with all the other sectors. The technology matrix A will be a matrix such that each column vector represents a different industry and each corresponding row vector represents what that industry inputs as a commodity into the column industry. The demand vector will be represented by D. The demand vector D is the amount of product the consumers will need. The total production vector X represents the total production that will be needed to satisfy the demand vector D. The total production vector X will be defined in this section.

Example 1 The technology matrix A below represents the relationships between the industries of Farming, Construction, and Clothing.

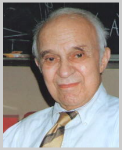
$$\begin{array}{cc} & \begin{array}{ccc} \text{Farming} & \text{Construction} & \text{Clothing} \end{array} \\ A = \begin{array}{l} \text{Farming} \\ \text{Construction} \\ \text{Clothing} \end{array} & \left(\begin{array}{ccc} .25 & .24 & .08 \\ .15 & .05 & .08 \\ .10 & .18 & .04 \end{array} \right) \end{array}$$

The relationships between the three industries in example one are as follows.

1. The entry a_{11} holds the number of units the farmer uses of his own product in producing one more unit of farming. The entry a_{21} holds the number of units the farmer needs of construction to produce one more unit of farming. The entry a_{31} holds the number of units the farmer needs of clothing to produce one more unit of farming.

2. The entry a_{12} holds the number of units that the builder needs from the farmer to produce one more unit of building .The entry a_{22} holds the number of units the builder needs of construction to produce one more unit of construction. The entry a_{32} holds the number of units the builder needs of clothing to produce one more unit of construction.

3. The entry a_{13} holds the number of units of farming that the tailor needs to produce one more unit of clothing. The entry a_{23} holds the number of units of construction that the tailor needs to produce one more unit of clothing. The entry a_{33} holds the number of units of clothing that the tailor needs to produce one more unit of his own product.



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$$A = \begin{array}{c} \\ \text{Farming} \\ \text{Construction} \\ \text{Clothing} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Farming} & \text{Construction} & \text{Clothing} \end{array} \\ \begin{pmatrix} 25/100 & 15/110 & 10/120 \\ 15/100 & 5/120 & 10/120 \\ 10/100 & 20/110 & 5/120 \end{pmatrix} \end{array}$$

- In general each entry in the technology matrix is represented as $a_{ij} = x_{ij}/x_j$, where x_j represents the physical output of sector j in our example the total production of an industry. Finally x_{ij} represents the amount of the product of sector i the row industry needed as input to sector j the column industry.

example 1, let's suppose a technology matrix

$$A = \begin{bmatrix} .25(100) & .24(110) & .08(120) \\ .15(100) & .05(110) & .08(120) \\ .10(100) & .18(110) & .04(120) \end{bmatrix}.$$

Suppose an external demand vector

$$D = \begin{bmatrix} 50 \\ 79.9 \\ 85.4 \end{bmatrix}.$$

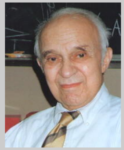
Suppose a total production vector

$$X = \begin{bmatrix} 100 \\ 110 \\ 120 \end{bmatrix}.$$

In the Argument that follows we will show that $D = X - AX$.

$$\begin{bmatrix} 50 \\ 79.9 \\ 85.4 \end{bmatrix} = \begin{bmatrix} 100 \\ 110 \\ 120 \end{bmatrix} - \begin{bmatrix} .25 & .24 & .08 \\ .15 & .05 & .08 \\ .10 & .18 & .04 \end{bmatrix} \begin{bmatrix} 100 \\ 110 \\ 120 \end{bmatrix}$$

To multiply A by X the jth element of X will be multiplied the the jth column of A. I have distributed the total production of the farming industry, the first element in X, through the farming industry, the first



column of A. The total production of the construction industry, the second element of X, was distributed through the construction industry, the second column of A. The total production of the clothing industry, the third element of X, was distributed through the clothing industry, the third column of A.

$$\begin{bmatrix} 50 \\ 79.9 \\ 85.4 \end{bmatrix} = \begin{bmatrix} 100 \\ 110 \\ 120 \end{bmatrix} - \begin{bmatrix} .25(100) & .24(110) & .08(120) \\ .15(100) & .05(110) & .08(120) \\ .10(100) & .18(110) & .04(120) \end{bmatrix}$$

Each element of AX is the output of an industry that is used in production. The farming industry produces 25 units for the production needs of itself, 15.4 units for the production needs of the construction industry and 9.6 units for the production needs of the clothing industry. The construction industry produces 15 units for the production needs of the farming industry, 5.5 units for its own production needs and 9.6 units for the production needs of the clothing industry.The clothing industry produces 10 units for the production needs of the farming industry, 19.8 units for the production needs of the construction industry and 4.8 units for the production needs of the clothing industry.

$$\begin{bmatrix} 50 \\ 79.9 \\ 85.4 \end{bmatrix} = \begin{bmatrix} 100 \\ 110 \\ 120 \end{bmatrix} - \begin{bmatrix} 25 + 15.4 + 9.6 \\ 15 + 5.5 + 9.6 \\ 10 + 19.8 + 4.8 \end{bmatrix}$$

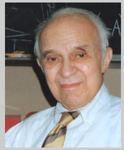
The total production the farming industry yields for all industries is 50 units.The total production the construction industry yields for all industries is 30.1 units.The total production the clothing industry yields for all industries is 34.6 units.

$$\begin{bmatrix} 50 \\ 79.9 \\ 85.4 \end{bmatrix} = \begin{bmatrix} 100 \\ 110 \\ 120 \end{bmatrix} - \begin{bmatrix} 50 \\ 30.1 \\ 34.6 \end{bmatrix}$$

Here we see that our demand vector D was in fact equal to our total production minus the production needed by all of the industries.

$$\begin{bmatrix} 50 \\ 79.9 \\ 85.4 \end{bmatrix} = \begin{bmatrix} 50 \\ 79.9 \\ 85.4 \end{bmatrix}$$

In conclusion we have shown that $D = X - AX$.



3. A Little Linear

X is the production vector needed to fill both the internal needs and the external demand. We start with $D = X - AX$. This means that our demand is equal to our total production minus the production needed by other industries as inputs, where total production X is the cumulative product made by each industry whether it is used in production or not. The production needed by other industries as inputs AX is the total amount of product that is used in production.

When making projections for the future you are not given the total production needed. The relations between industries, the technology matrix A, is known and so is the demand for each industry D. Our goal would be to find the total production that will be needed to fill a certain demand. We must solve the equation $D = X - AX$ for X.

Our initial equation is

$$D = X - AX.$$

Any matrix multiplied by an identity matrix is equal to itself $IX = X$. Therefore we can replace X with IX,

$$D = IX - AX.$$

We factor out an X from both terms on the right side of the equation. It is important to factor out the X to the right because if it's factored out to the left matrix multiplication will break down when multiplying the demand vector D on the left side by $(I - A)^{-1}$.

$$D = (I - A)X$$

In order to solve for X we multiply by $(I - A)^{-1}$ on the left side of both sides of the equation.

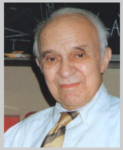
$$(I - A)^{-1}D = (I - A)^{-1}(I - A)X.$$

Any matrix multiplied by it's inverse is equal to the identity matrix $(I - A)^{-1}(I - A) = I$. Substituting I for $(I - A)^{-1}(I - A)$ we get

$$(I - A)^{-1}D = IX.$$

Since IX=X as stated before we substitute X for IX, $(I - A)^{-1}D = X$ With a little rearranging we have our equation to solve for the total production needed to satisfy an economy with a known demand vector D and a known technology matrix A.

$$X = (I - A)^{-1}D$$



$$X = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} .25 & .24 & .08 \\ .15 & .05 & .08 \\ .10 & .18 & .04 \end{bmatrix} \right)^{-1} \begin{bmatrix} 50 \\ 79.9 \\ 85.4 \end{bmatrix} = \begin{bmatrix} 100 \\ 110 \\ 120 \end{bmatrix}$$

4. 1992 benchmark input-output (I-O) accounts

The I-O accounts or input-output accounts are the data sets used in input-output analysis. The five I-O accounts can be used to show the production of commodities (goods and services) by each industry, the use of commodities by each industry, the commodity composition of the gross domestic product(GDP), and the industry of value added. For this project I used two of the five I-O accounts, the Make table, and commodity-by-industry direct requirements table. In this paper the Make table is refereed to as the demand vector .The commodity-by-industry direct requirements table is referred to as matrix A, or the technology coefficients matrix.

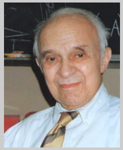
The I-O classification system is primarily based on the Standard Industrial Classification (SIC) system, which classifies establishments into industries on the basis of the primary activities of the establishment. The economy separated into 97 I-O industries from the 498-industry details. A sector in one of these I-O industries.

The benchmark I-O accounts are based primarily on data collected from the economic censuses conducted every 5 years by the Bureau of the Census. The data collected in 1997 should be available later this December. I will be using the demand and technical coefficient matrix as an analytical tool since they show the interdependence among the producers and consumers in the economy. We will be able to estimate the direct and indirect effects of change in final uses on industries and commodities.

5. Code for GDP Program

First I get a sum for the initial economy's total demand, and a sum for the changed economy's total demand. The initial economy's total demand is the sum of each element the the initial demand vector. The changed economy's demand is the new demand vector, to get the sum of this we will also add up all of it's elements.

```
for m=1:2
    Total_Old_Demand=Total_Old_Demand+Old_Economy_Demand(m,1);
end
```



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```
Total_Old_Demand
for m=1:2
    Total_New_Demand=Total_New_Demand+New_Economy_Demand(m,1);
end
```

Here the equation $X = (I - A)^{-1}D$ is used to compute the initial production needed to satisfy the initial demand. Also the new total production needed to satisfy the new demand.

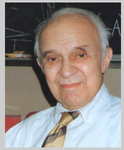
```
Old_Production=inv(eye(97)-A)*Old_Economy_Demand; Old_Production
for m=1:97
    Total_Old_Production=Total_Old_Production+Old_Production(m,1);
end
Total_Old_Production
New_Production=inv(eye(97)-A)*D;
for m=1:97
    Total_New_Production=Total_New_Production+New_Production(m,1);
end
Total_New_Production
```

Here I subtract the initial production from the new production. Then I add up the differences. The sum of the differences represents the amount of production that was added to the economy because of the changes in demand.

```
New_Production
for m=1:97
    Added_Production(m,1)=New_Production(m,1)-Old_Production(m,1);
end
Added_Production
for m=1:97
    Total_Growth=Total_Growth+ Added_Production(m,1);
end
Total_Growth
```

To calculate the change in GDP I divide the production that was added to the economy by the total production of the initial economy then multiply by 100 to get the percent change. This percent change in the production of the economy is the change in the Gross Domestic Product or GDP.

```
Percent_Change_in_GDP=(Total_Growth/Total_Old_Production)*100
```



6. Analyzing the GDP

The GDP or the Gross Domestic Product is the total amount of production produced in the United States of America. Equally weighted demand vectors will effects the GDP differently.

The technology matrix we will use is for the 1992 U.S. economy and is 97×97, each column represents a different industry. I used five different examples all starting with a GDP of 219775812.49 million. I added 1000 million to ten different sectors. I used four different methods to pick the ten sectors.

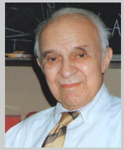
7. Value Added and other Demand Vectors

Value added consists of three components-compensation of employees, indirect business tax and non-tax liability, and "other value added"

1. Most Value Added

Sector No.	**Industry Name**
4	Agriculture, Forestry, and Fishery Services
9&10	Nonmetallic Minerals Mining
15	Tobacco Products
26a	Newspapers and Periodicals
65a	Railroads and their services, Ground Transportation
68a	Electric Services
69a	Wholesale Trade
69b	Retail Trade
71a	Owner-Occupied Dwellings
71b	Real Estate and Royalties

Sectors chosen were, 4, 9 and 10, 15, 26 a, 65 a, 68 a, 69a, 69b, 71 a and 71 b. The total increase of 10000 million equally distributed amongst the demand vectors added 122709.53 million to the GDP causing a .14 percent increase.



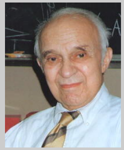
2. Least Value Added

Sector No.	Industry Name
1	Livestock and Their Products
14	Food and Kindred Products
25	Cardboard Containers and Boxes
28	Plastics and Synthetic Materials
31	Petroleum Refining and Related Products
39	Metal Containers
51	Computer and Office Equiptment
59a	Motor Vehicles
59b	Truck and Bus Bodies, Trailers and Parts
68b	Gas Production and Distribution

Sectors chosen were, 1, 14, 25, 28, 31, 39, 51, 59 a, 59 b and 68 b. The total increase of 10000 million equally distributed amongst the demand vectors added 94712.11 million to the GDP causing a .11 percent increase.

3. The computer chose these ten randomly.

Sector No.	Industry Name
2	Other Agricultural Products
25	Cardboard Containers and Boxes
43	Engines and Turbines
47	Metal Working Machinery and Equiptment
49	General Industry Machinery and Equiptment
54	Household Appliances
65d	Air Transportation
69a	Wholesale Trade
73b	Legal, Engineering, Accounting and Related Services
79	State and Local Government



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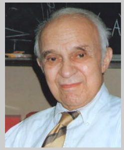
Sectors chosen were, 2, 25, 43, 47, 49, 54, 65 d, 69 a, 73 b and 79. The total increase of 10000 million equally distributed amongst the demand vectors added 56166.84 million to the GDP causing a .06 percent increase.

4. These ten were chosen by number only by myself.

Sector No.	Industry Name
1	Livestock and Their Products
5&6	Metallic Ore Mining
7	Coal Mining
9&10	Nonmetallic Minerals Mining
22&23	Furniture and Fixture
44&45	farming, Construction and Mining Machinery
68a	Electric Services
73b	Legal, Engineering and Accounting Services
75	Automotive Repair and Services
78	Federal Government Expenditures

Sectors chosen were, 1, 5 and 6, 7, 9 and 10, 22 and 23, 44 and 45, 68 a, 73b, 75,also 78. The total increase of 10000 million equally distributed amongst the demand vectors added 1027085.67 million to the GDP causing a 1.15 percent increase.

5. Ten students during the presentation chose one sector each knowing the industries that they picked.



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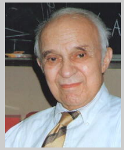
Sector No.	Industry Name
3	Forestry and Fishery Products
31	Petroleum Refining and Related Products
43	Engines and Turbines
60	Aircraft Parts
62	Scientific and Controlling Instruments
65d	Air Transportation
67	Radio and TV Broadcasting
70a	Finance
73d	Advertising
77b	Education and Social Services

Sectors chosen were, 3, 31, 43, 60, 62, 65d, 67 , 70a , 73d and 77b. The total increase of 10000 million equally distributed amongst the demand vectors added 2286437.06 million to the GDP causing a 1.04 percent increase.

8. Conclusion

It is obvious that the change in GDP is dependent on which sectors you choose. Each different demand vector that was created gave dramatically different increases in the GDP ranging from .06 to 1.15 percent. Input-Output analysis is a valuable tool for GDP projections into the future. This application of linear algebra is much more heavy on data analysis than on the math it's self. When using Leontief open production model in input-output analysis it is crucial to also have some general understanding of economics.



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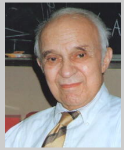
[6] My fellow students from Dave Arnold's Fall of 2001 Linear Algerbra class for all of their help with both linear algebra and computer questions.



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