Iterative Relocation of Earthquake Epicenters

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Abstract

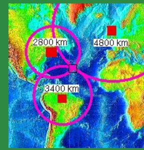
The time for a seismic wave to travel from earthquake source to receiver can be expressed as a function of the path the energy followed, a model of the subsurface structure, and an initial guess at the hypocentral parameters, location and origin time. When this travel time is expressed for the myriad of observation stations, the resultant equations can be expressed by the matrix equation:

$$\triangle d = G \triangle m$$

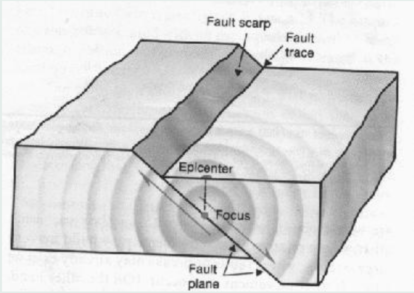
where $\triangle d$ is a residual time vector, $\triangle m$ is a model perturbation vector, and G is a matrix containing partial derivatives associated with each element of $\triangle m$. In other words, there is a relationship between the residual travel time and the adjustment of the original estimate as to the three dimensional location for an earthquake (related by G.) To solve this matrix equation for $\triangle m$, we are finding a perturbation to the model, allowing an adjustment of the original guess. This process is iteratively repeated until refinement of the model is negligible. To operate on G, a single value decomposition is used to find a psuedo-inverse, or least squares analysis is used. To better understand the error involved in this procedure, the computation of a covariance matrix can be utilized. A more extensive application of this problem includes analysis of velocity changes of seismic waves, which provides an indirect observation of the material composing the local subsurface.

1. Background

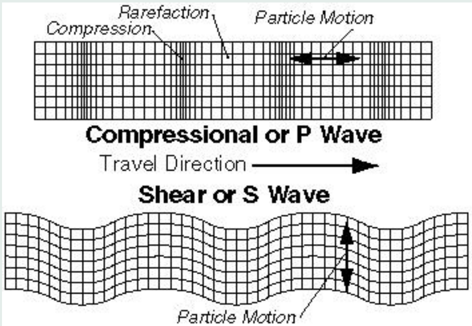
Earthquakes are a common occurrence around the world, which provides an excellent opportunity for geologists to study them. It is well known that there is a higher frequency of earthquakes in some regions versus others, but the precise reasons for why this discrepancy exists is still being studied. Earthquakes are a result of a pressure release as the earth shifts its crust, according to plate tectonics. When two



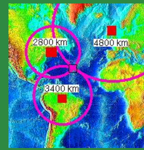
plates interact with one another, a certain amount of shear stress can be tolerated by the geologic material that composes the subsurface. Once this range of tolerance is surpassed, an earthquake occurs, releasing a tremendous amount of energy. Due to the nature of how energy is released in this setting, some distinct aspects of the energy can be observed. Through compression of subsurface blocks during an earthquake, a resultant compressional wave is emitted (a pulse, or P-wave) which has a fluctuation of energy parallel to its axis of propagation.



Through shearing of subsurface blocks during an earthquake, a wave of energy is emitted (a shear, or S-wave) that has a fluctuation of energy perpendicular to its axis of propagation.

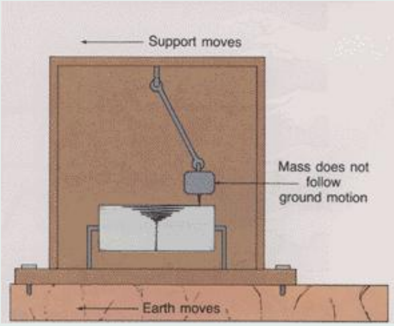


The intrinsic nature of these two waves results in P-waves traveling faster than S-waves, which will be important in the latter portions of this study. Also included in the emission of energy are surface

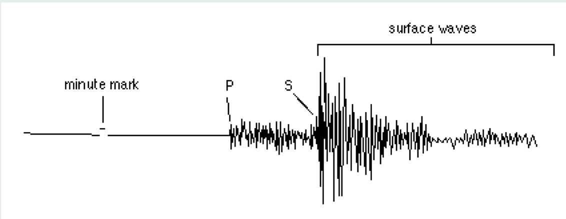


waves, due to a projection of the energy released to the surface of the earth. For reference, we are most sensitive to surface waves (intense shaking), somewhat sensitive to S-waves ("rolling" motion), and weakly sensitive to P-waves (pulsating or sporadic.) The neat thing that can be perceived here is when one can feel all three types of waves, it is obvious which waves travel the fastest (P, S, then surface.)

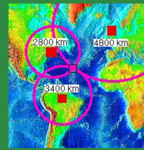
Observations made concerning earthquakes are simply interpretations of the shaking that occurs when energy passes through the earth. To make such observations, the seismograph



is used. This instrument is fixed to the rigid bedrock of the region (compared to loose sediment) where the seismograph will shake along with the earth. Also involved is a weighted pen which resists the shaking of the seismograph, and is set up to write continuously on a piece of paper. This piece of paper is known as a seismogram.



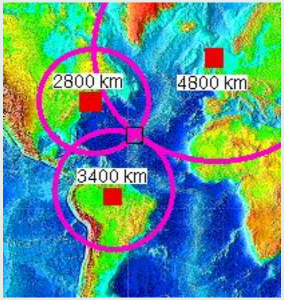
By considering the relative speeds of each of the energy waves, it can be determined which wave is



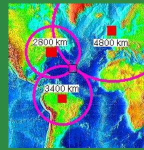
represented on the seismogram, but more importantly, the wave can be assigned a specific arrival time. Once identifying the arrival times for both the P and S waves and comparing the traveling velocities of each of the waves, the distance from the source of the waves (hypocenter) to the receiving seismograph(the observation station, or OBS) can be inferred. By considering these values, we can begin to study and understand earthquakes.

Once we begin to understand earthquakes, we can piece together vital information about the regions in which the earthquake occurs. By comparing similar information from adjacent areas, regions can be interpreted (which is vital in the study of geology.) The items being understood here are the three dimensional structure of the earth, something that cannot be directly observed at this point in time (the deepest drilling holes only scratch the surface.) If the arrival times of the earthquake waves can be compared with the location of the hypocenter and the corresponding velocities of energy waves, we have the possibility of understanding what the earthquake has to reveal.

To accurately consider observations of a release of seismic energy, a sufficient number of seismographs must be utilized. By having multiple observation stations, the error associated with interpreting the hypocentral location is minimized. Also, when attempting to find a radius of where the earthquake occurred with respect to a given seismograph, the collaborative radii of all the observation stations will provide a more accurate approximation of the epicenter (surface projection of hypocenter.)



To get started, we need to understand the geometry involved in the relationship between the hypocenter and an arbitrary OBS. We establish a distance vector which will be crucial to our problem, but by defining the vector as linear path, a large assumption of a constant-velocity for the waves to travel by is introduced (more on that later.) To establish our problem, consideration of P-wave arrival time, an initial guess of the hypocenter location, an origin time of the event, OBS locations, and P-wave travel



velocities are necessary. The units of these data values are seconds after a reference time, kilometers with respect to a defined origin (hypocenter location & OBS locations), seconds, and km/s respectively. The hypocenter location and origin time will define a vector, m_0, which is a key quantity in the problem. The values in kilometers will have been converted from degrees latitude and longitude, using the appropriate conversion for the location on the globe. To organize all of this information, the data for the latitude, longitude, elevations of the OBS, and P-wave arrival times will be stored in separate vectors (with their appropriate units) [Giardini]. By organizing the information this way, we can use MATLAB to run an efficient iteration. The main goal for this problem is to re-locate the original model parameters (m_0) until an acceptable value is found (discussed later.) But we first need to set up the problem so we can get to this point.

2. Defining the problem

When considering arrival time data, it is obvious that the observed time of arrival for an energy wave is the time that the earthquake occurred (t_0) plus the time the energy took to travel. When considering the geometry and the assumption that the P-wave travels at a constant velocity ($v_p = 6.1\,\mathrm{km/s}$), the travel time for a P-wave from its source to an arbitrary receiver is given by the following equation:

$$t_i = \frac{\sqrt{(x_i - ex)^2 + (y_i - ey)^2 + (z_i - ez)^2}}{v_p} \tag{1}$$

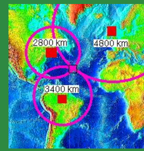
and when combined with the origin time of the earthquake (et_0) provides the relationship between observed arrival times and our unknown variables:

$$P_{\text{arrival}} = t_i + et_0 \tag{2}$$

Before we move on with the problem, it is important to consider the assumptions we have made, as well as the possibilities of not making such assumptions. By not assuming a linear ray path, we can account for the fact that the earth's subsurface is not a continuous medium. When we define the path of the waves to conform to a heterogeneous medium, a "slowness" integral is utilized:

$$t = \int u(r)\, ds$$

where r is a position vector, ds is the incremental path length, and $u(r)$ is a slowness model, indicating wave travel velocity at the position vector [Toomey]. Once we apply the problem under these conditions,



the change in velocities when approaching the OBS can infer the medium that the waves are traveling through. This is a very powerful application of this problem, especially when considering the difficulty and/or inaccuracy of other methods of identifying the precise local material composing the subsurface. Another assumption made (in our presentation and MATLAB program) is a constant hypocentral depth. This assumption is not as significant as the velocity assumption due to the abilitiy to extend the two dimensional version of the problem into three dimensions. For our purposes here, the problem will be discussed with variable parameters for x, y, and z directions.

For starters, we need to define the variables involved in Equation 1. We let ex_0, ey_0, and ez_0 define our original hypocentral estimate. Similarly, et_0 will define our earthquake event time. These values involve the information desired in our problem, and together will be called m_0. To establish a relationship between m_0 and our arrival times, we can refer to Equation 2, which gives us arrival time, t_i, as a function of four variables.

$$t_i = T_i(ex, ey, ez, et)$$

3. Linearizing/Establishing Matrix Equation

Because the vector defining our relative positions of the hypocenter and receivers contains a non-linear expression, we need to linearize the arrival time equation so we can proceed. To do so, we will use Taylor's Theorem to expand (2), and drop all terms that are greater than first order:

$$t_i = T_i(ex_0, ey_0, ez_0, et_0) + \left.\frac{\partial T_i}{\partial ex}\right|_{m_0}(ex - ex_0) + \left.\frac{\partial T_i}{\partial ey}\right|_{m_0}(ey - ey_0) + \left.\frac{\partial T_i}{\partial ez}\right|_{m_0}(ez - ez_0) + (et - et_0)$$

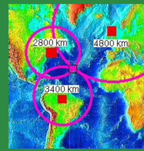
where we have expanded about m_0.

Moving the term $T_i(ex_0, ey_0, ez_0, et_0)$ to the left hand side of (2), and defining the resultant difference $P_{\text{arrival}} - T_i$ as d_i, we get the following relationship which provides a relationship between the time residual d_i and the change in the model vector entries ex_0, etc:

$$d_i = \left.\frac{\partial T_i}{\partial ex}\right|_{m_0}(ex - ex_0) + \left.\frac{\partial T_i}{\partial ey}\right|_{m_0}(ey - ey_0) + \left.\frac{\partial T_i}{\partial ez}\right|_{m_0}(ez - ez_0) + (et - et_0)$$

This expression yields a relationship for each of our observation station, an most importantly, is linear! When considering this equation for all OBS, we can build a matrix equation that considers the whole-scale relationship between the change in the hypocentral parameters and the residual in time related to our P_{arrival} data:

$$\triangle d = G\triangle m$$



where $\triangle d$ contains the time residuals $(n \times 1)$, G contains the partial derivatives involved with Taylor's $(n \times 4)$ and $\triangle m$ contains the hypocentral parameter changes (4×1). Upon solving this matrix equation for $\triangle m$, we find an adjustment to our original model of the hypocentral parameters, m_0, which ideally gives a better estimate of the hypocenter location and earthquake origin time. It should be noted here that performing the previously described calculation can lead to an estimate that is less accurate than the initial estimate, m_0. In such a case, the problem diverges from the true answer! This situation is not very common, and for our example, was not the case. To solve our matrix equation, we can use a couple of techniques: calculating the psuedo-inverse of G, or using least squares analysis. We will use the psuedo-inverse to proceed.

4. Solving for $\triangle m$

Our matrix G contains a large number of entries, leading to an expensive computation. To begin solving for our model vector $\triangle m$, we manipulate the relationship between Singular Value Decomposition and the pseudo-inverse. This technique diagonalizes G, allowing for a solution to our matrix equation, while simultaneously reducing the expense of the calculations.

The goal of SVD is to diagonalize our matrix G, factoring it into three matrices, U, Σ , and V. This diagonalization involves choosing orthogonal bases for the $C(G^T)$, entries become our V matrix, and for the $C(G)$, entries become our U matrix. We make these into orthonormal base vectors. These U and V matrices yield our Σ matrix (a diagonal composed of single value entries.)

The Σ matrix, containing our singular values, σ_i, is very similar to our Λ matrix consisting of eigenvalues, λ_i. The difference comes about from using orthonormal bases vectors to build our U and V matrices.

The orthonormal bases vectors that build V must satisfy the following relationship

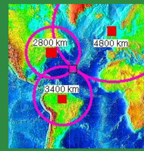
$$Gv_1 \perp \cdots \perp Gv_i \perp \cdots \perp Gv_r$$

We can now find unit bases vectors that comprise the $C(A^T)$

$$\hat{v}_i = \frac{Gv_i}{\|Gv_i\|}$$

The G/V relationship yields the U and Σ matrices as such, diagonalizing G

$$GV = U\Sigma$$



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$$G\begin{bmatrix} v_1 & \cdots & v_i & \cdots & v_r \end{bmatrix} = \begin{bmatrix} \sigma_1 u_1 & \cdots & \sigma_i u_i & \cdots & \sigma_r u_r \end{bmatrix}$$

The $U\Sigma$ matrix factors into

$$\begin{bmatrix} u_1 & \cdots & u_i & \cdots & u_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_i & & \\ & & & \ddots & \\ & & & & \sigma_r \end{bmatrix}$$

At this point we find it important to explain where the remaining U and V vectors disappeared to. When we use the pseudo-inverse later, all the following $(u_{r+1} \cdots u_m)$ and $(v_{r+1} \cdots v_n)$ entries become zero due to the following properties:

$$G^{\dagger} u_i = \frac{1}{\sigma_i} v_i, \text{ for } \quad i \leqslant r$$

and

$$G^{\dagger} u_i = 0, \text{for} \quad i > r$$

The pseudo-inverse sends the u vectors $(u_{r+1}, \ldots, u_m)$ in $N(G^T)$ to zero, and performs a similar operation for the v vectors as well. (This will come to light during the upcoming discussion of pseudo-inverses.)

We just used SVD to diagonalize our matrix G

$$GV = U\Sigma.$$

Since U and V are orthogonal matrices, the bases vectors in their columns are orthonormal. Thus,

$$V^T V = I \qquad \text{and} \qquad V^T = V^{-1}.$$

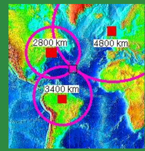
Similarly,

$$U^T U = I \qquad \text{and} \qquad U^T = U^{-1}.$$

So, $GV = U\Sigma$ can be written as

$$G = U\Sigma V^{-1} \Rightarrow G = U\Sigma V^T$$

To solve our system for $\triangle m$, we must have an invertible G matrix. The typical method for solving an "unsolvable" system of equations is the least squares approximation. This is a tedious, drawn out



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process. Naturally, this led us to seek out a better method. After many semesters with our professor Dave Arnold, we inevitably repeat the 'Dave-ism', "The important thing is to get the answer, but after doing all the tedious, drawn out work, you've got to ask yourself, is there a better way?"

Yes, there is a better way. It's called the pseudo-inverse. This exciting method allows us to invert any matrix (even the so-called non-invertible matrices.) This concept allows a whole new approach for matrix operations. The conditionality of 'if the inverse exists' imposed upon us is now trivial.

The pseudo-inverse takes our matrix G and manipulates it into an invertible matrix $G^\dagger$ (where $\dagger$ represents pseudo-inverse notation) such that

$$G = U\Sigma V^T \Rightarrow G^T = U\Sigma^\dagger V^T$$

The pseudo-inverse of our diagonalized matrix $\Sigma^\dagger$ involves transposing Σ (yielding the original Σ matrix), then taking the reciprocal of each singular value entry (the σ_i which lie on the main diagonal.)

Since we can now invert every matrix, we can solve any system. This is basically the least squares method (only infintesimally simpler.)

We solve our original system:

$$\triangle d = G\triangle m$$

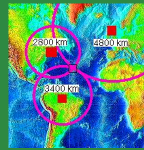
for $\triangle m$ by first diagonalizing G using SVD

$$G = U\Sigma V^T.$$

Then, using the pseudo-inverse, solving our system for $\triangle m$:

$$\begin{aligned} G^\dagger &= U\Sigma^\dagger V^T \\ \triangle d &= G\triangle m \\ G^\dagger\triangle d &= G^\dagger G\triangle m \\ G^\dagger\triangle d &= \triangle m \\ \therefore \triangle m &= G^\dagger\triangle d. \end{aligned}$$

Now that we can solve for $\triangle m$, we must determine the significance of our result. Finding a large change in our model vector warrants the desire to re-assess our problem, or perform another iteration. In such a case, we would simply apply our result, $\triangle m$, to our original hypocenter model, m_0, then use this modified model to perform the next iteration. Once we repeat these iterations such that the result is acceptable, we can be assured that this method is exhausted. To determine whether or not our



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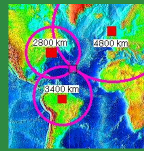
results are acceptable, we can employ a couple of different methods. A qualitative approach we can take is the use of MATLAB to plot the location of the hypocentral estimates after each iteration. Once a convergence of the changes to the hypocenter is established, there will be no change in consecutive plots. The disadvantage here is that we do not have a good idea of the error involved. To better understand the error involved, we can use a covariance matrix. The covariance expresses the relationship between two variables...defined as the average of the sum of the products of the deviation scores for all the objects [Reyment/Jöreskog]. The values contained within the covariance matrix will approach 1 as the hypocentral parameters become increasingly accurate. To compute the covariance matrix, multiply the result $\triangle m$ by its transpose. While there are some minor adjustments to this result, $\triangle m \triangle m^T$ is directly proportional to the covariance matrix. One other method of selecting a stopping point for the iterations is to use the F test [DeGroot]. Because of the previous two methods we did not choose to investigate the F test. At this point, we have located the hypocenter and the origin time of the earthquake, and have exhausted the ability of our method.

5. Final Remarks

Ultimately, geologists use this problem to find an accurate location of a hypocenter as well as to decipher the subsurface of the area local to the earthquake. These quantities of information are vital to the study of geology, and more specifically, seismology. By having accurate assessments of the hypocenter, a seismologist can provide a more accurate interpretation of the mechanism that caused the earthquake, which is crucial when considering that many interpretations are based on inferences. As stated earlier, the ability to identify the subsurface material is of high importance due to the fact that the existing material is directly related to the geologic processes responsible for the present structures. Although we did not apply the rigor of the tomography (subsurface interpretation) to our problem, we did consider a three dimensional approach. When our version of the problem is extended to consider tomography, our results will naturally change according to the actual velocities of P-wave propagation, yielding more accurate results.

Being able to determine the mechanisms behind earthquakes and to relate the mechanisms to the local subsurface, seismologists have a powerful tool with which to study the earth. The theory of Plate Tectonics is still in the midst of a rapid development, and tools like the process described in this paper (amongst many others) will provide a much broader understanding in the future. By having a better understanding of the processes of the earth, humans can prepare themselves better according to the regions they live in.



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6. Acknowledgements

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