

The Mathematics of GPS through Linear Algebra

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Abstract

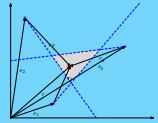
This article will be introducing the mathematics calculating the accuracy of Global Positioning Systems (GPS) from N passive sensors to a point of interest.

1. Introduction

The Global Positioning System (GPS) is the trend of today. From military to civilian use, the idea of knowing your location accurately on earth is important in many fields. But, how does GPS relay accurate coordinates for the position of a missile or your position while you drive your car? Many variables can cause error (such as atmospheric interference) with calculating precise coordinates.

The GPS should use an active device to accurately pinpoint the position of choice. For a GPS receiver to work properly, the receiver needs to be in contact with at least three active orbiting satellites at one time. In order for a GPS receiver to be active, effective communication between satellites and the receiver are necessary to pinpoint the coordinates of your choice. More specifically, the time it takes for a satellite signal to reach the GPS receiver will reduce the error with the coordinates reading the receiver calculated (this is signal propagation delay). But, how accurate would GPS be if a passive receiver were used?

A passive receiver, as opposed to an active receiver, will receive information from satellites concerning their position in relation to the receiver's position just as an active device would, but there would be no active communication between the receiver and satellites. This is referred to as "spoofing." The receiver would still relay coordinates, but how accurate would those coordinates be? This question has been labeled as the N-Site problem. This is where our work comes in. In this paper we will develop a mathematical formula for the position of an object viewed from three passive sensors.



[Introduction](#)

[Expanding the Problem](#)

[Mathematical . . .](#)

[Sample Calculation](#)

[Conclusion](#)

[Home Page](#)

[Title Page](#)



[Page 1 of 14](#)

[Go Back](#)

[Full Screen](#)

[Close](#)

[Quit](#)

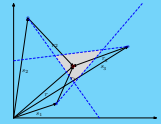
2. Expanding the Problem

The first step with developing the N-Site solution is to analyze the geometry of the situation. At this time we know the direction from N passive sensors to a point of interest (in this paper we will use $N=3$). In a perfect scenario, the lines of sight from the three satellites will all intersect at the point of interest, providing the object's position, but in real life the lines will miss each other, and overlap in such a way that a "gray" area exists. The calculation for the point of interest within the gray area can be found using triangulation. Our mathematical model is a sample of using triangulation (our point of interest exists within a triangle, and it's position can be found using information from the lines forming the triangle). Although this problem exists in three space, we will use a two-dimensional model with vectors to demonstrate our N-Site solution. The reason why a 2D model can be an example for a realistic 3D model is because vectors will be used. Otherwise, if coordinates and angles were used, we would be restricted to a dimensional model only.

In [Figure 1](#) we will visualize a scenario with three passive sensors (in blue) and one point of interest (in red). We put in an origin to help simplify our explanations. Vectors \mathbf{s}_1 through \mathbf{s}_3 are vectors extended from the origin to the three passive sensors. In general this vector will be noted at \mathbf{s}_i . Vector \mathbf{r} extends from the origin to the point of interest. Vector \mathbf{r} minimizes the mean square error within the gray triangle area because the vector points directly to the point of interest. Vectors \mathbf{r}_1 through \mathbf{r}_3 extend from each sensor to the point of interest. The lines of sight of the sensors are denoted with the blue dashed lines. The area where the blue lines intersect the "gray" triangle with the red point of interest in the middle. Our goal, more specifically, is to minimize the mean square error within the triangle to find the point of interest mathematically in vector notation.

[Figure 2](#) is a more simplified version of [Figure 1](#) and will be the direct model of our Mathematical Development section. It simply involves one sensor and the point of interest. The vectors are labelled accordingly as with the above figure except for the introduction of a new vector, \mathbf{d}_1 . This unit vector extends in the direction of the line of sight. For general purposes we will denote the unit vector as \mathbf{d}_i .

Now, in the Mathematical Development section we have the freedom to develop the formula in two or three dimensions. We will develop the formula in three dimensions but restrict our sample calculation with two dimensions for simplicity. At this point we are ready to proceed with mathematical development as shown below.



[Introduction](#)

[Expanding the Problem](#)

[Mathematical . . .](#)

[Sample Calculation](#)

[Conclusion](#)

[Home Page](#)

[Title Page](#)

[◀](#) [▶](#)

[◀](#) [▶](#)

[Page 2 of 14](#)

[Go Back](#)

[Full Screen](#)

[Close](#)

[Quit](#)

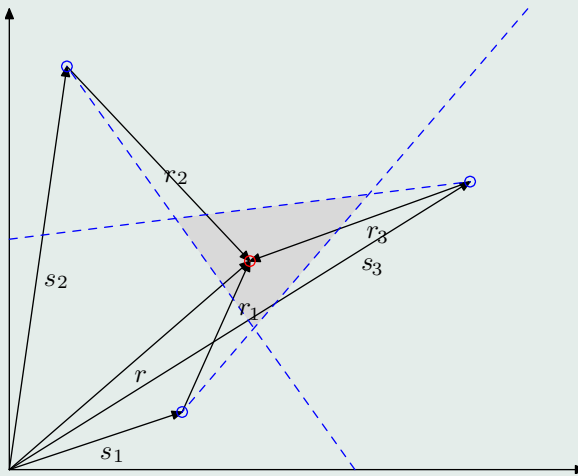


Figure 1: Three passive sensors detecting a point of interest.

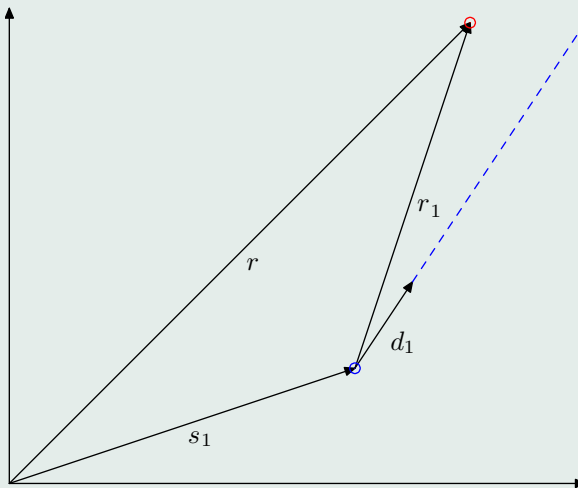
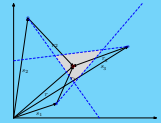


Figure 2: A scaled version of **Figure 1** with one sensor.



[Introduction](#)

[Expanding the Problem](#)

[Mathematical ...](#)

[Sample Calculation](#)

[Conclusion](#)

[Home Page](#)

[Title Page](#)



Page 3 of 14

[Go Back](#)

[Full Screen](#)

[Close](#)

[Quit](#)

3. Mathematical Development

First we will find the length of the projection vector by projecting \mathbf{r} onto \mathbf{d}_i as represented in the equation below. The output is the scalar p_i which is also equal to $\mathbf{r}_i^T \mathbf{d}_i$.

$$p_i = \frac{\mathbf{r}_i^T \mathbf{d}_i}{\mathbf{d}_i^T \mathbf{d}_i} = \mathbf{r}_i^T \mathbf{d}_i$$

The mean squared error E^2 is shown below. Note the mean squared error equals the sum of all the errors magnitude squared from N sensors used. Also note that vector \mathbf{r} minimizes the mean square error.

$$E^2 = \frac{1}{N} \sum_{i=1}^N \|\mathbf{r}_i - p_i \mathbf{d}_i\|^2$$

$$E^2 = \frac{1}{N} \sum_{i=1}^N (\mathbf{r}_i - p_i \mathbf{d}_i)^T (\mathbf{r}_i - p_i \mathbf{d}_i)$$

$$E^2 = \frac{1}{N} \sum_{i=1}^N (\mathbf{r}_i^T - p_i \mathbf{d}_i^T) (\mathbf{r}_i - p_i \mathbf{d}_i)$$

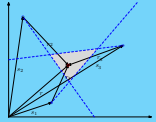
Expanding the two equations with multiplication provides the following line.

$$E^2 = \frac{1}{N} \sum_{i=1}^N (\mathbf{r}_i^T \mathbf{r}_i - \mathbf{r}_i^T p_i \mathbf{d}_i - \mathbf{r}_i p_i \mathbf{d}_i^T + p_i^2 \mathbf{d}_i^T \mathbf{d}_i)$$

$$E^2 = \frac{1}{N} \sum_{i=1}^N (\mathbf{r}_i^T \mathbf{r}_i - 2p_i \mathbf{r}_i^T \mathbf{d}_i + p_i^2)$$

Remember that $p_i = \mathbf{r}_i^T \mathbf{d}_i$ from the first line. Now we can substitute p_i for $\mathbf{r}_i^T \mathbf{d}_i$ so we can simplify the following equation as shown below.

$$E^2 = \frac{1}{N} \sum_{i=1}^N (\mathbf{r}_i^T \mathbf{r}_i - 2p_i^2 + p_i^2)$$



Introduction

Expanding the Problem

Mathematical ...

Sample Calculation

Conclusion

Home Page

Title Page



Page 4 of 14

Go Back

Full Screen

Close

Quit

$$E^2 = \frac{1}{N} \sum_{i=1}^N \mathbf{r}_i^T \mathbf{r}_i - p_i^2$$

To apply more simplification we now have the following equation.

$$E^2 = \frac{1}{N} \sum_{i=1}^N \|\mathbf{r}_i\|^2 - p_i^2$$

Now applying that $\mathbf{r}_i = \mathbf{r} - \mathbf{s}_i$ through geometry as shown in **Figure 2**,

$$E^2 = \frac{1}{N} \sum_{i=1}^N \|\mathbf{r} - \mathbf{s}_i\|^2 - [(\mathbf{r} - \mathbf{s}_i)^T \mathbf{d}_i]^2.$$

$$E^2 = \frac{1}{N} \sum_{i=1}^N (\mathbf{r} - \mathbf{s}_i)^T (\mathbf{r} - \mathbf{s}_i) - [(\mathbf{r}^T - \mathbf{s}_i^T) \mathbf{d}_i]^2$$

When $(\mathbf{r} - \mathbf{s}_i)^T (\mathbf{r} - \mathbf{s}_i)$ is expanded and $[(\mathbf{r}^T - \mathbf{s}_i^T) \mathbf{d}_i]$ is distributed the equation becomes

$$E^2 = \frac{1}{N} \sum_{i=1}^N \mathbf{r}^T \mathbf{r} - 2\mathbf{r}^T \mathbf{s}_i + \mathbf{s}_i^T \mathbf{s}_i - (\mathbf{r}^T \mathbf{d}_i)^2 + 2\mathbf{r}^T \mathbf{d}_i \mathbf{s}_i^T \mathbf{d}_i - (\mathbf{s}_i^T \mathbf{d}_i)^2.$$

Since $\mathbf{s}_i^T \mathbf{s}_i - (\mathbf{s}_i^T \mathbf{d}_i)^2 = \mathbf{s}_i^T (\mathbf{s}_i - (\mathbf{s}_i^T \mathbf{d}_i) \mathbf{d}_i)$ the equation now becomes

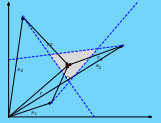
$$E^2 = \frac{1}{N} \sum_{i=1}^N \mathbf{r}^T \mathbf{r} - 2\mathbf{r}^T \mathbf{s}_i - (\mathbf{r}^T \mathbf{d}_i)^2 + 2\mathbf{r}^T \mathbf{d}_i \mathbf{s}_i^T \mathbf{d}_i + \mathbf{s}_i^T (\mathbf{s}_i - (\mathbf{s}_i^T \mathbf{d}_i) \mathbf{d}_i).$$

By subtracting \mathbf{s}_i from the projection of \mathbf{s}_i onto \mathbf{d}_i the resulting vector is orthogonal to \mathbf{s}_i so $\mathbf{s}_i^T (\mathbf{s}_i - (\mathbf{s}_i^T \mathbf{d}_i) \mathbf{d}_i)$ can be dropped. We now have

$$E^2 = \frac{1}{N} \sum_{i=1}^N \mathbf{r}^T \mathbf{r} - 2\mathbf{r}^T \mathbf{s}_i - (\mathbf{r}^T \mathbf{d}_i)^2 + 2\mathbf{r}^T \mathbf{d}_i \mathbf{s}_i^T \mathbf{d}_i.$$

To locate \mathbf{r} where E^2 is at a minimum, we take the gradient vector of E^2 and set it equal to zero. We will take the gradient of each part of the above equation to show derivation. First we will take the gradient of $\mathbf{r}^T \mathbf{r}$.

$$\nabla_{\mathbf{r}} \mathbf{r}^T \mathbf{r} = \nabla_{\mathbf{r}} \mathbf{r} \cdot \mathbf{r}$$



Introduction

Expanding the Problem

Mathematical . . .

Sample Calculation

Conclusion

Home Page

Title Page



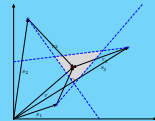
Page 5 of 14

Go Back

Full Screen

Close

Quit



$$\nabla_r \mathbf{r}^T \mathbf{r} = \nabla_r |\mathbf{r}|^2$$

$$\nabla_r \mathbf{r}^T \mathbf{r} = 2|\mathbf{r}|\nabla_r |\mathbf{r}|$$

$$\nabla_r \mathbf{r}^T \mathbf{r} = 2|\mathbf{r}|\frac{\mathbf{r}}{|\mathbf{r}|}$$

$$\nabla_r \mathbf{r}^T \mathbf{r} = 2\mathbf{r}$$

Now we will take the gradient of $-2\mathbf{r}^T \mathbf{s}_i$.

$$\nabla_r (-2\mathbf{r}^T \mathbf{s}_i) = -2\nabla_r \mathbf{r}^T \mathbf{s}_i$$

$$\nabla_r (-2\mathbf{r}^T \mathbf{s}_i) = -2\nabla_r \mathbf{r} \cdot \mathbf{s}_i$$

If we let $\mathbf{r} = \langle r_1, r_2, r_3 \rangle$ and $\mathbf{s}_i = \langle s_1, s_2, s_3 \rangle$ the equation now becomes

$$\nabla_r (-2\mathbf{r}^T \mathbf{s}_i) = -2\nabla_r \langle r_1 s_1 + r_2 s_2 + r_3 s_3 \rangle .$$

$$\nabla_r (-2\mathbf{r}^T \mathbf{s}_i) = -2 \langle s_1, s_2, s_3 \rangle$$

$$\nabla_r (-2\mathbf{r}^T \mathbf{s}_i) = -2\mathbf{s}_i$$

Now we will take the gradient of $-(\mathbf{r}^T \mathbf{d}_i)^2$.

$$\nabla_r (-(\mathbf{r}^T \mathbf{d}_i)^2) = -\nabla_r (\mathbf{r}^T \mathbf{d}_i)^2$$

$$\nabla_r (-(\mathbf{r}^T \mathbf{d}_i)^2) = -\nabla_r (\mathbf{r} \cdot \mathbf{d}_i)^2$$

$$\nabla_r (-(\mathbf{r}^T \mathbf{d}_i)^2) = -2(\mathbf{r} \cdot \mathbf{d}_i)\nabla_r (\mathbf{r} \cdot \mathbf{d}_i)$$

If we let $\mathbf{r} = \langle r_1, r_2, r_3 \rangle$ and $\mathbf{d}_i = \langle d_1, d_2, d_3 \rangle$ and $\nabla_r (\mathbf{r} \cdot \mathbf{d}_i) = \langle d_1, d_2, d_3 \rangle$ the equation now becomes

$$\nabla_r (-(\mathbf{r}^T \mathbf{d}_i)^2) = -2(\mathbf{r} \cdot \mathbf{d}_i)\mathbf{d}_i .$$

$$\nabla_r (-(\mathbf{r}^T \mathbf{d}_i)^2) = -2(\mathbf{r}^T \mathbf{d}_i)\mathbf{d}_i$$

Finally we will take the gradient of $2\mathbf{r}^T \mathbf{d}_i \mathbf{s}_i^T \mathbf{d}_i$.

$$\nabla_r 2\mathbf{r}^T \mathbf{d}_i \mathbf{s}_i^T \mathbf{d}_i = 2\nabla_r \mathbf{r}^T \mathbf{d}_i \mathbf{s}_i^T \mathbf{d}_i$$

$$\nabla_r 2\mathbf{r}^T \mathbf{d}_i \mathbf{s}_i^T \mathbf{d}_i = 2(\mathbf{s}_i^T \mathbf{d}_i)\nabla_r (\mathbf{r}^T \mathbf{d}_i)$$

Introduction

Expanding the Problem

Mathematical . . .

Sample Calculation

Conclusion

Home Page

Title Page



Page 6 of 14

Go Back

Full Screen

Close

Quit

Knowing that $\nabla_r(\mathbf{r}^T \mathbf{d}_i) = \mathbf{d}_i$ we can proceed with

$$\nabla_r 2\mathbf{r}^T \mathbf{d}_i \mathbf{s}_i^T \mathbf{d}_i = 2(\mathbf{s}_i^T \mathbf{d}_i) \mathbf{d}_i.$$

So now we can conclude with the the gradient that

$$\nabla_r E^2 = \frac{1}{N} \sum_{i=1}^N 2\mathbf{r} - 2\mathbf{s}_i - 2(\mathbf{r}^T \mathbf{d}_i) \mathbf{d}_i + 2(\mathbf{s}_i^T \mathbf{d}_i) \mathbf{d}_i = 0.$$

In the last equation we set the gradient equal to zero. The purpose of this is to find the location of \mathbf{r} for where E^2 is at a minimum.

For simplification we will have new notations for the two projections onto \mathbf{d}_i . One projection is the projection of vector \mathbf{r} onto \mathbf{d}_i and the other projection is the i^{th} site vector \mathbf{s}_i onto \mathbf{d}_i . So $(\mathbf{r}^T \mathbf{d}_i) \mathbf{d}_i$ is now $m_i \mathbf{d}_i$ and $(\mathbf{s}_i^T \mathbf{d}_i) \mathbf{d}_i$ is now $n_i \mathbf{d}_i$. Our more simplified equation is now

$$\frac{1}{N} \sum_{i=1}^N 2\mathbf{r} - 2\mathbf{s}_i - 2m_i \mathbf{d}_i + 2n_i \mathbf{d}_i = 0.$$

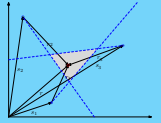
Now what we want to do is group the \mathbf{r} terms together and leave all other terms on the other side of the equal sign. So we have

$$\frac{1}{N} \sum_{i=1}^N 2\mathbf{r} - 2m_i \mathbf{d}_i = \frac{1}{N} \sum_{i=1}^N 2\mathbf{s}_i - 2n_i \mathbf{d}_i.$$

To simplify the equation we will multiply both sides by $\frac{1}{2}N$ to make the equation

$$\sum_{i=1}^N \mathbf{r} - m_i \mathbf{d}_i = \sum_{i=1}^N \mathbf{s}_i - n_i \mathbf{d}_i.$$

Figure 3 shows the vectors $\mathbf{r} - m_i \mathbf{d}_i$ and $\mathbf{s}_i - n_i \mathbf{d}_i$. We moved the \mathbf{d}_i vector to the origin and extended the lines of sight from the vector. The vector $\mathbf{r} - m_i \mathbf{d}_i$ is not only the minimum distance from the origin-based line of sight to the point of interest, it's also orthogonal to the line of sight. The geometry is also similar with the $\mathbf{s}_i - n_i \mathbf{d}_i$ vector. It is the minimum length from the origin-based line of sight to the sensor. It's also orthogonal to the line of sight. From the derived mathematics above we can conclude that $\mathbf{r} - m_i \mathbf{d}_i = \mathbf{s}_i - n_i \mathbf{d}_i$ for all i . Again, in an ideal situation, the lines of sight will all



Introduction

Expanding the Problem

Mathematical . . .

Sample Calculation

Conclusion

Home Page

Title Page



Page 7 of 14

Go Back

Full Screen

Close

Quit

intersect at the point of interest (the tip of \mathbf{r}) but since the lines of sight don't intersect, error will have to be included in the problem. But, let's look at the geometry again.

As **Figure 3** shows, the difference between $\mathbf{s}_i - n_i \mathbf{d}_i$ and $\mathbf{r} - m_i \mathbf{d}_i$ equals the orthogonal displacement between \mathbf{r} and the i^{th} line of sight. So, how do you compensate for that? Knowing that we need at least three sensors in a two dimensional calculation to find the point of interest, we will reason that the errors between one point of interest and the line of sight in magnitude equals the other inconsistencies with other lines of sight in relation to the point of interest in magnitude. We are making a big assumption here that there is no error in our calculations. In our sample calculation the sensors in relation to the lines of sight are close to each other so error isn't a problem in this case. But, we can infer that there could be large values of error with the point of interest being very far away from the sensors. Here there could be a big difference between $\mathbf{s}_i - n_i \mathbf{d}_i$ and $\mathbf{r} - m_i \mathbf{d}_i$ so there would be a large angular difference between \mathbf{r} and \mathbf{d}_i .

At this point we will just continue with the \mathbf{r} side of the equation and find a convenient form of this equation for numeral calculations. We will develop a matrix for this. Now $\mathbf{r} - m_i \mathbf{d}_i$ can be cast in the form of $\mathbf{P}\mathbf{r} = \mathbf{u}$. The origins of $\mathbf{P}\mathbf{r} = \mathbf{u}$ come from the following diagram. The vector \mathbf{r} is represented by \mathbf{r} . The variable \mathbf{u} represents the differences between the line of sight and the point of interest. \mathbf{P} is found by expanding $m_i \mathbf{d}_i$ assuming $\mathbf{r} = \langle r_x, r_y, r_z \rangle$ and $\mathbf{d}_i = \langle d_{ix}, d_{iy}, d_{iz} \rangle$ in matrix form. Remember, you can use two or three dimensional vectors! But, the dimension you choose should be used throughout the derivation. We now have

$$m_i \mathbf{d}_i = (\mathbf{r}^T \mathbf{d}_i) \mathbf{d}_i = (r_x d_{ix} + r_y d_{iy} + r_z d_{iz}) \mathbf{d}_i.$$

Now to apply the matrix form of the previous equation remembering that $\mathbf{d}_i = \langle d_{ix}, d_{iy}, d_{iz} \rangle$.

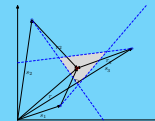
$$m_i \mathbf{d}_i = (r_x d_{ix} + r_y d_{iy} + r_z d_{iz}) \begin{bmatrix} d_{ix} \\ d_{iy} \\ d_{iz} \end{bmatrix} = \begin{bmatrix} r_x d_{ix}^2 + r_y d_{ix} d_{iy} + r_z d_{ix} d_{iz} \\ r_x d_{ix} d_{iy} + r_y d_{iy}^2 + r_z d_{iy} d_{iz} \\ r_x d_{ix} d_{iz} + r_y d_{iy} d_{iz} + r_z d_{iz}^2 \end{bmatrix}.$$

This matrix equals

$$m_i \mathbf{d}_i = \begin{bmatrix} d_{ix}^2 & d_{ix} d_{iy} & d_{ix} d_{iz} \\ d_{ix} d_{iy} & d_{iy}^2 & d_{iy} d_{iz} \\ d_{ix} d_{iz} & d_{iy} d_{iz} & d_{iz}^2 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}.$$

So

$$m_i \mathbf{d}_i = [\mathbf{d}_i \mathbf{d}_i^T] \mathbf{r}.$$



[Introduction](#)

[Expanding the Problem](#)

[Mathematical...](#)

[Sample Calculation](#)

[Conclusion](#)

[Home Page](#)

[Title Page](#)

[◀](#) [▶](#)

[◀](#) [▶](#)

Page 8 of 14

[Go Back](#)

[Full Screen](#)

[Close](#)

[Quit](#)

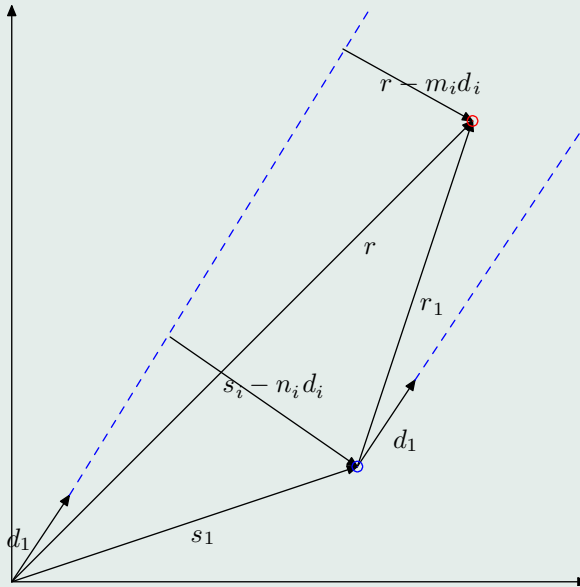
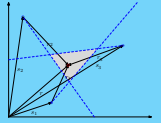


Figure 3: The geometry of the line of sight onto the point of interest and the line of sight onto the sensor.



- [Introduction](#)
- [Expanding the Problem](#)
- [Mathematical . . .](#)
- [Sample Calculation](#)
- [Conclusion](#)

[Home Page](#)

[Title Page](#)

◀◀ ▶▶

◀ ▶

Page 9 of 14

[Go Back](#)

[Full Screen](#)

[Close](#)

[Quit](#)

We have shown that $m_i \mathbf{d}_i$ is the outer product of the i^{th} line of sight unit vector and \mathbf{r} . So

$$\mathbf{r} - m_i \mathbf{d}_i = I_3 \mathbf{r} - (\mathbf{d}_i \mathbf{d}_i^T) \mathbf{r}.$$

$$\mathbf{r} - m_i \mathbf{d}_i = (I_3 - \mathbf{d}_i \mathbf{d}_i^T) \mathbf{r}$$

I_3 is a 3×3 identity matrix. In two dimensions it would be a 2×2 identity matrix. So our previous equation

$$\sum_{i=1}^N \mathbf{r} - m_i \mathbf{d}_i = \sum_{i=1}^N \mathbf{s}_i - n_i \mathbf{d}_i$$

now becomes

$$\sum_{i=1}^N [I_3 - \mathbf{d}_i \mathbf{d}_i^T] \mathbf{r} = \sum_{i=1}^N \mathbf{s}_i - n_i \mathbf{d}_i.$$

This is in the form $\mathbf{P}\mathbf{r} = \mathbf{u}$ where $\mathbf{P} = \sum_{i=1}^N I_3 - \mathbf{d}_i \mathbf{d}_i^T$ and $\mathbf{u} = \sum_{i=1}^N \mathbf{s}_i - n_i \mathbf{d}_i$. Now to solve for the minimum mean-square error, the solution is simply done by multiplying both sides of $\mathbf{P}\mathbf{r} = \mathbf{u}$ by \mathbf{P}^{-1} . So $\mathbf{r} = \mathbf{P}^{-1}\mathbf{u}$. Now we will provide a sample calculation to show just how this works.

4. Sample Calculation

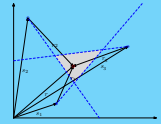
We will now look at a sample calculation in a 2-dimensional perspective. The positions of the satellites were chosen and vectors were drawn to those positions and are called $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$. The unit vectors of the lines of sight were then found and are given the labels of $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3$.

$$\mathbf{s}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} 1 \\ 7 \end{bmatrix}, \mathbf{s}_3 = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

$$\mathbf{d}_1 = \frac{1}{\sqrt{85}} \begin{bmatrix} 6 \\ 7 \end{bmatrix}, \mathbf{d}_2 = \frac{1}{\sqrt{74}} \begin{bmatrix} 5 \\ -7 \end{bmatrix}, \mathbf{d}_3 = \frac{1}{\sqrt{65}} \begin{bmatrix} -8 \\ -1 \end{bmatrix}$$

All of the vectors were then graphed with the lines of sight drawn in blue. **Figure 4** is the graph that shows the vectors, the line of sight, and the general area where the point of interest will be found.

We can now apply our vectors to locate \mathbf{P} and \mathbf{u} . This is easily done with some vector and matrix multiplication along with vector and matrix addition/subtraction.



Introduction

Expanding the Problem

Mathematical...

Sample Calculation

Conclusion

Home Page

Title Page



Page 10 of 14

Go Back

Full Screen

Close

Quit

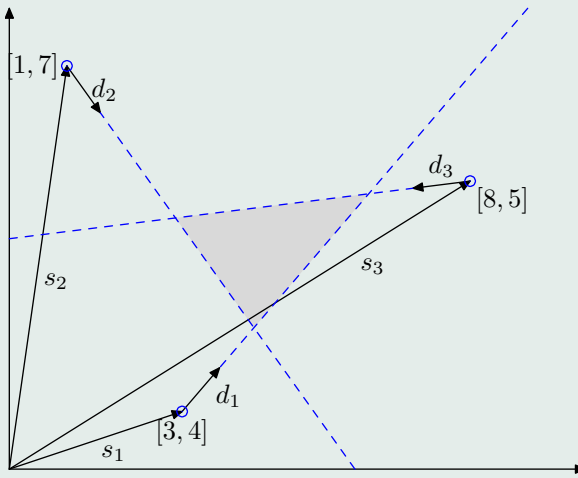


Figure 4: The gray triangle encompassed by the lines of sight of the three sensors.

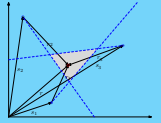
$$\mathbf{P} = \sum_{i=1}^N \mathbf{I}_2 - \mathbf{d}_i \mathbf{d}_i^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{85} \begin{bmatrix} 36 & 42 \\ 42 & 49 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{74} \begin{bmatrix} 25 & -35 \\ -35 & 49 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{65} \begin{bmatrix} 64 & 8 \\ 8 & 1 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \frac{1}{85} \begin{bmatrix} 36 & 42 \\ 42 & 49 \end{bmatrix} - \frac{1}{74} \begin{bmatrix} 25 & -35 \\ -35 & 49 \end{bmatrix} - \frac{1}{65} \begin{bmatrix} 64 & 8 \\ 8 & 1 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 2029/1618 & -151/1047 \\ -151/1047 & 2825/1618 \end{bmatrix}$$

$$\mathbf{P} \approx \begin{bmatrix} 1.254 & -0.1442 \\ -0.1442 & 1.746 \end{bmatrix}$$

$$\mathbf{u} = \sum_{i=1}^N \mathbf{s}_i - n_i \mathbf{d}_i = \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \frac{25}{85} \begin{bmatrix} 6 \\ 7 \end{bmatrix} + \begin{bmatrix} 1 \\ 7 \end{bmatrix} + \frac{44}{74} \begin{bmatrix} 5 \\ -7 \end{bmatrix} + \begin{bmatrix} 8 \\ 5 \end{bmatrix} + \frac{69}{65} \begin{bmatrix} -8 \\ -1 \end{bmatrix}$$



Introduction

Expanding the Problem

Mathematical ...

Sample Calculation

Conclusion

Home Page

Title Page

◀ ▶

◀ ▶

Page 11 of 14

Go Back

Full Screen

Close

Quit

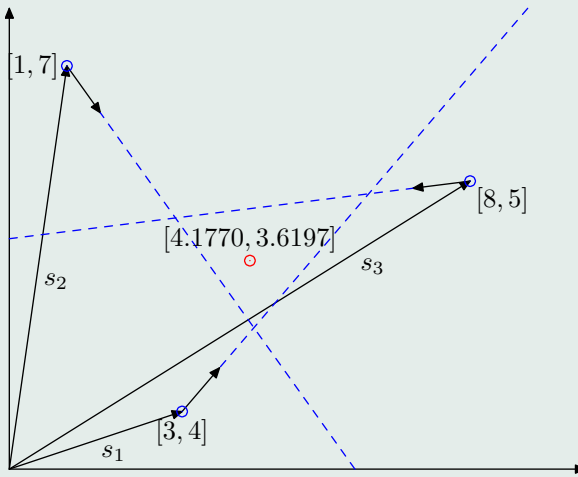


Figure 5: Our solution fits inside the gray triangle. We have minimized the mean square error within the triangle.

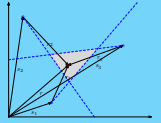
$$\mathbf{u} = \begin{bmatrix} 3221/683 \\ 1538/269 \end{bmatrix}$$

$$\mathbf{u} \approx \begin{bmatrix} 4.716 \\ 5.718 \end{bmatrix}$$

Now that we have \mathbf{P} and \mathbf{u} we can use this to solve the equation $\mathbf{r} = \mathbf{P}^{-1}\mathbf{u}$

$$\mathbf{r} = \begin{bmatrix} 1.254 & -.1442 \\ -.1442 & 1.746 \end{bmatrix}^{-1} \begin{bmatrix} 4.1770 \\ 3.6197 \end{bmatrix}$$

Thus by looking at [Figure 5](#) we have determined the position of \mathbf{r} which is found by minimizing the error of the lines of sight around the point of interest.



[Introduction](#)

[Expanding the Problem](#)

[Mathematical...](#)

[Sample Calculation](#)

[Conclusion](#)

[Home Page](#)

[Title Page](#)

[◀](#) [▶](#)

[◀](#) [▶](#)

Page 12 of 14

[Go Back](#)

[Full Screen](#)

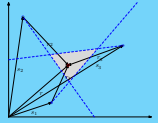
[Close](#)

[Quit](#)

5. Conclusion

As you have just seen, we have successfully located a point of interest with three passive sensors via triangulation. This paper is one sample of the many applications of linear algebra. The power of linear algebra allowed us to take a relatively difficult task and make it very user friendly. Through the geometry of the sensors and lines of sight we were able to even assume that error could be ignored.

Global Positioning Systems are an important tool of today. The ability of determining the position of a missile or your position in the middle of nowhere is a valuable resource. We have demonstrated not only that the concept of GPS is relatively simple, but the geometry is flexible for any geographical situation. Whether spoofing is on or off, GPS's are relatively accurate in any case.



[Introduction](#)

[Expanding the Problem](#)

[Mathematical . . .](#)

[Sample Calculation](#)

[Conclusion](#)

[Home Page](#)

[Title Page](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 13 of 14

[Go Back](#)

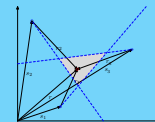
[Full Screen](#)

[Close](#)

[Quit](#)

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- [2] Bonser, Gordon. For his help and knowledge of GPS.
- [3] Draayer, Bret. “N-Site Insights.” **The College Mathematics Journal**. vol. 31 no. 4 Sept 2000 250-258.
- [4] Urbach, Rey. For his help and knowledge of GPS.
- [5] We would like to thank Doug, Dave, Adam, and Maurissa for their help and proofreading skills.



Introduction

Expanding the Problem

Mathematical . . .

Sample Calculation

Conclusion

Home Page

Title Page



Page 14 of 14

Go Back

Full Screen

Close

Quit