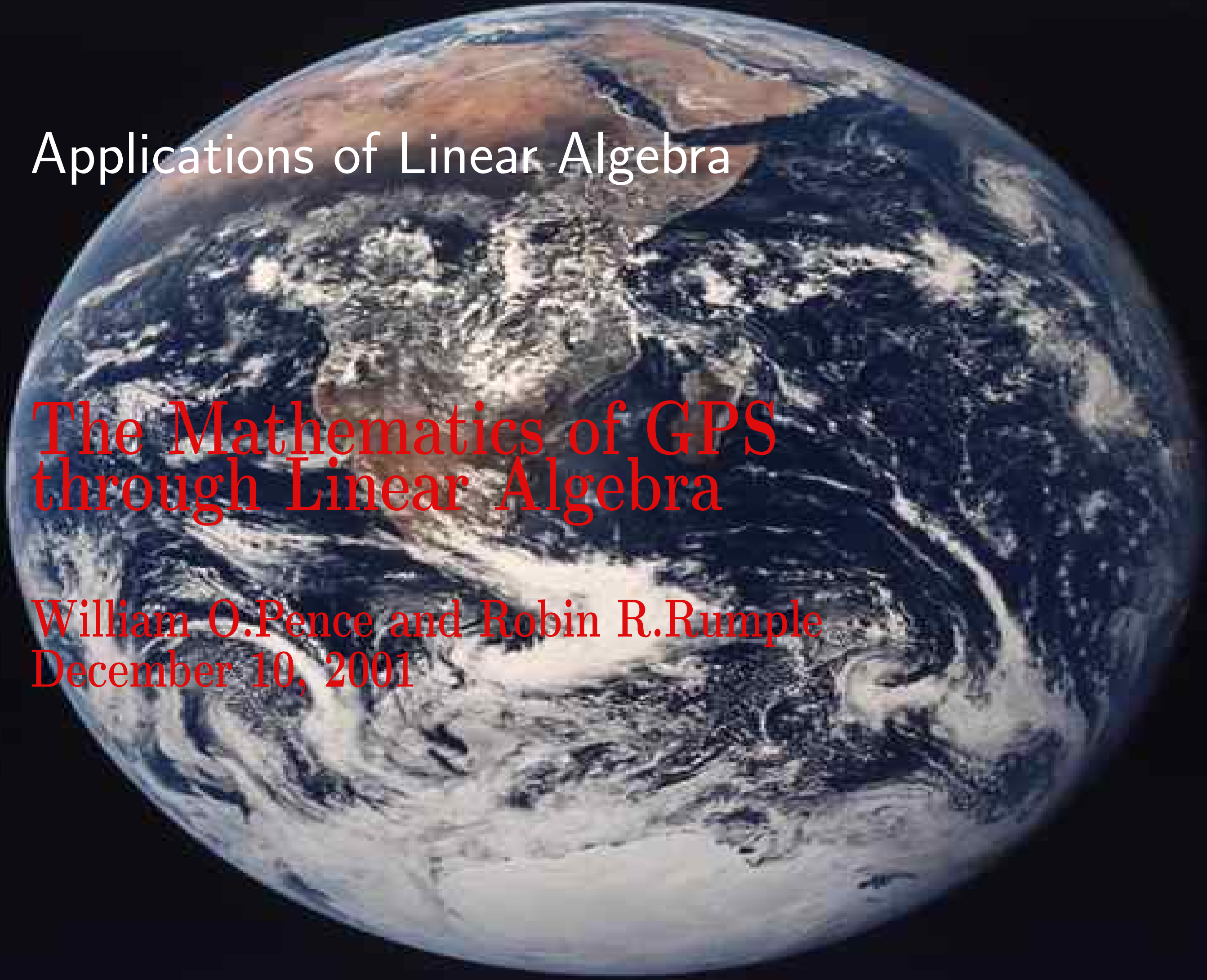

Applications of Linear Algebra

The Mathematics of GPS through Linear Algebra

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Introduction

- In this presentation we will develop a mathematical formula for locating a point of interest from N passive sensors.

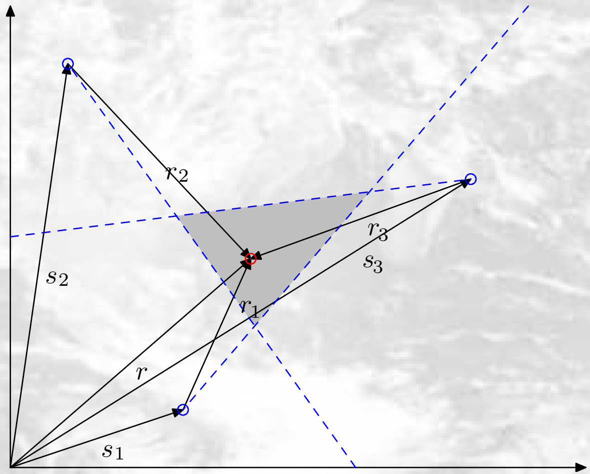

Figure 1: Three passive sensors detecting a point of interest.

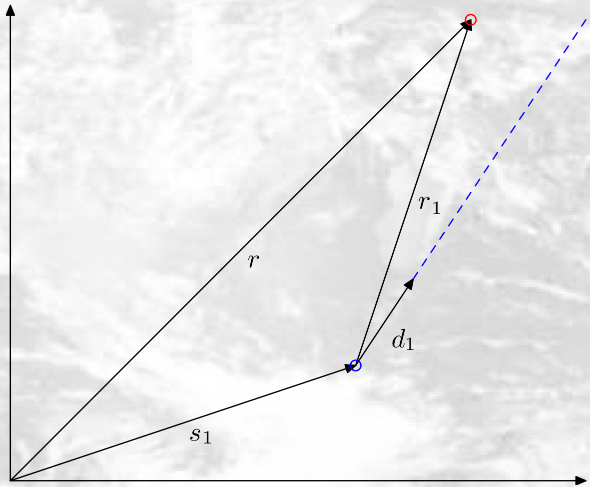

Figure 2: A scaled version of Figure 1 with one sensor.

Mathematical Development

- Using the definition of projection we know that.

$$p_i = \frac{\mathbf{r}_i^T \mathbf{d}_i}{\mathbf{d}_i^T \mathbf{d}_i} = \mathbf{r}_i^T \mathbf{d}_i$$

- We can then find the error($E = \mathbf{r}_i - p_i\mathbf{d}_i$). We then square the error to find the over all error made by all the satellites, which minimizes the gray triangle to the point of interest.

$$E^2 = \frac{1}{N}\sum_{i=1}^{N} ||\mathbf{r}_i - p_i\mathbf{d}_i||^2$$

$$E^2 = \frac{1}{N}\sum_{i=1}^{N} (\mathbf{r}_i - p_i\mathbf{d}_i)^T(\mathbf{r}_i - p_i\mathbf{d}_i)$$

$$E^2 = \frac{1}{N}\sum_{i=1}^{N} (\mathbf{r}_i^T\mathbf{r}_i - 2p_i\mathbf{r}_i^T\mathbf{d}_i + p_i^2)$$

Remember that

$$p_i = \mathbf{r}_i^T\mathbf{d}_i$$

- Now the equation becomes a little more simplified due to the fact that $-2p_i\mathbf{r}_i^T\mathbf{d}_i$ equals $-2{p_i}^2$.

$$E^2 = \frac{1}{N}\sum_{i=1}^{N}\mathbf{r}_i^T\mathbf{r}_i - p_i^2$$

One can take the $\mathbf{r}_i^T\mathbf{r}_i$ and change it into $||\mathbf{r}_i||^2$.

$$E^2 = \frac{1}{N}\sum_{i=1}^{N}||\mathbf{r}_i||^2 - p_i^2$$

- Now we can use $\mathbf{r}_i = \mathbf{r} - \mathbf{s}_i$ in the equation $E^2 = \frac{1}{N}\sum_{i=1}^{N} ||\mathbf{r}_i - p_i\mathbf{d}_i||^2$.

$$E^2 = \frac{1}{N}\sum_{i=1}^{N} ||\mathbf{r} - \mathbf{s}_i||^2 - [(\mathbf{r} - \mathbf{s}_i)^T\mathbf{d}_i]^2$$

$$E^2 = \frac{1}{N}\sum_{i=1}^{N} (\mathbf{r} - \mathbf{s}_i)^T(\mathbf{r} - \mathbf{s}_i) - [(\mathbf{r}^T - \mathbf{s}_i^T)\mathbf{d}_i]^2$$

- When FOILed the equation becomes

$$E^2 = \frac{1}{N}\sum_{i=1}^{N} \mathbf{r}^T\mathbf{r} - 2\mathbf{r}^T\mathbf{s}_i - (\mathbf{r}^T\mathbf{d}_i)^2 + 2\mathbf{r}^T\mathbf{d}_i\mathbf{s}_i^T\mathbf{d}_i + \mathbf{s}_i^T(\mathbf{s}_i - (\mathbf{s}_i^T\mathbf{d}_i)\mathbf{d}_i)$$

- By subtracting $\mathbf{s}_i$ from the projection of $\mathbf{s}_i$ onto $\mathbf{d}_i$ the resulting vector is orthogonal to $\mathbf{s}_i$ so $\mathbf{s}_i^T(\mathbf{s}_i - (\mathbf{s}_i^T\mathbf{d}_i)\mathbf{d}_i)$ can be dropped. We now have

$$E^2 = \frac{1}{N}\sum_{i=1}^{N} \mathbf{r}^T\mathbf{r} - 2\mathbf{r}^T\mathbf{s}_i - (\mathbf{r}^T\mathbf{d}_i)^2 + 2\mathbf{r}^T\mathbf{d}_i\mathbf{s}_i^T\mathbf{d}_i$$

- To locate $\mathbf{r}$ where E^2 is at a minimum, we take the gradient vector of E^2 and set it equal to zero. We will take the gradient of each part of the above equation to show derivation.

$$\nabla_r E^2 = \frac{1}{N}\sum_{i=1}^{N} 2\mathbf{r} - 2\mathbf{s}_i - 2(\mathbf{r}^T\mathbf{d}_i)\mathbf{d}_i + 2(\mathbf{s}_i^T\mathbf{d}_i)\mathbf{d}_i = 0$$

- For simplification we will have new notations for the two projections onto $\mathbf{d}_i$. One projection is the projection of vector $\mathbf{r}$ onto $\mathbf{d}_i$ and the other projection is the i^{th} site vector $\mathbf{s}_i$ onto $\mathbf{d}_i$. So $(\mathbf{r}^T\mathbf{d}_i)\mathbf{d}_i$ is now $m_i\mathbf{d}_i$ and $(\mathbf{s}_i^T\mathbf{d}_i)\mathbf{d}_i$ is now $n_i\mathbf{d}_i$.

- Our more simplified equation is now

$$\frac{1}{N}\sum_{i=1}^{N} 2\mathbf{r} - 2\mathbf{s}_i - 2m_i\mathbf{d}_i + 2n_i\mathbf{d}_i = 0$$

- Now what we want to do is group the $\mathbf{r}$ terms together and leave all other terms on the other side of the equal sign and multiplying both by $\frac{1}{2}N$.

$$\sum_{i=1}^{N} \mathbf{r} - m_i\mathbf{d}_i = \sum_{i=1}^{N} \mathbf{s}_i - n_i\mathbf{d}_i$$

- At this point we will just continue with the $\mathbf{r}$ side of the equation and find a convenient form of this equation for number crunching. We will develop a matrix for this.

- Now $\mathbf{r} - m_i\mathbf{d}_i$ can be cast in the form of $\mathbf{P} * \mathbf{r} = \mathbf{u}$. The origins of $\mathbf{P} * \mathbf{r} = \mathbf{u}$ come from the following diagram.

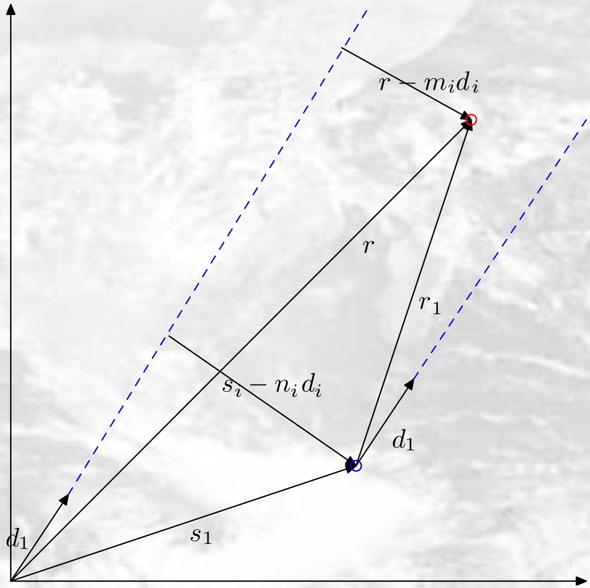

Figure 3: The geometry of line of sight onto the point and the line of sight onto the sensor.

- The vector **r** is represented by **r**. The variable **u** represents the differences between the line of sight and the point of interest. **P** is found by expanding $m_i\mathbf{d}_i$ assuming **r** $=< r_x, r_y, r_z >$ and $\mathbf{d}_i =< d_{ix}, d_{iy}, d_{iz} >$ which gives

$$m_i\mathbf{d}_i = (\mathbf{r}^T\mathbf{d}_i)\mathbf{d}_i = (r_x d_{ix} + r_y d_{iy} + r_z d_{iz})\mathbf{d}_i$$

$$m_i\mathbf{d}_i = \begin{bmatrix} d_{ix}^2 & d_{ix}d_{iy} & d_{ix}d_{iz} \\ d_{ix}d_{iy} & d_{iy}^2 & d_{iy}d_{iz} \\ d_{ix}d_{iz} & d_{iy}d_{iz} & d_{iz}^2 \end{bmatrix} * \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}.$$

$$m_i\mathbf{d}_i = [\mathbf{d}_i\mathbf{d}_i^T] \cdot \mathbf{r}$$

- This shows that $m_i\mathbf{d}_i$ is the outer product of the i^{th} line of sight unit vector and $\mathbf{r}$. So

$$\mathbf{r} - m_i\mathbf{d}_i = I_3 * \mathbf{r} - (\mathbf{d}_i\mathbf{d}_i^T) * \mathbf{r}$$

$$\mathbf{r} - m_i\mathbf{d}_i = (I_3 - \mathbf{d}_i\mathbf{d}_i^T) * \mathbf{r}$$

where I_3 is a 3×3 identity matrix. So our previous equation

$$\sum_{i=1}^{N} \mathbf{r} - m_i\mathbf{d}_i = \sum_{i=1}^{N} \mathbf{s}_i - n_i\mathbf{d}_i$$

now becomes

$$\sum_{i=1}^{N} [I_3 - \mathbf{d}_i\mathbf{d}_i^T] * \mathbf{r} = \sum_{i=1}^{N} \mathbf{s}_i - n_i\mathbf{d}_i$$

- This is in the form $\mathbf{P} * \mathbf{r} = \mathbf{u}$ where $\mathbf{P} = \sum_{i=1}^{N} I_3 - \mathbf{d}_i \mathbf{d}_i^T$ and $\mathbf{u} = \sum_{i=1}^{N} \mathbf{s}_i - n_i \mathbf{d}_i$. Now to solve for the minimum mean-square error, the solution is simply done by multiplying both sides of $\mathbf{P} * \mathbf{r} = \mathbf{u}$ by $\mathbf{P}^{-1}$. So $\mathbf{r} = \mathbf{P}^{-1} * \mathbf{u}$. Onto a sample calculation to show just how this works.

Sample Calculation

- We will now look at a sample calculation.

$$\mathbf{s}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} 1 \\ 7 \end{bmatrix}, \mathbf{s}_3 = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

$$\mathbf{d}_1 = \frac{1}{\sqrt{85}} \begin{bmatrix} 6 \\ 7 \end{bmatrix}, \mathbf{d}_2 = \frac{1}{\sqrt{74}} \begin{bmatrix} 5 \\ -7 \end{bmatrix}, \mathbf{d}_3 = \frac{1}{\sqrt{65}} \begin{bmatrix} -8 \\ -1 \end{bmatrix}$$

- Figure 4 shows the vectors and the line of site.

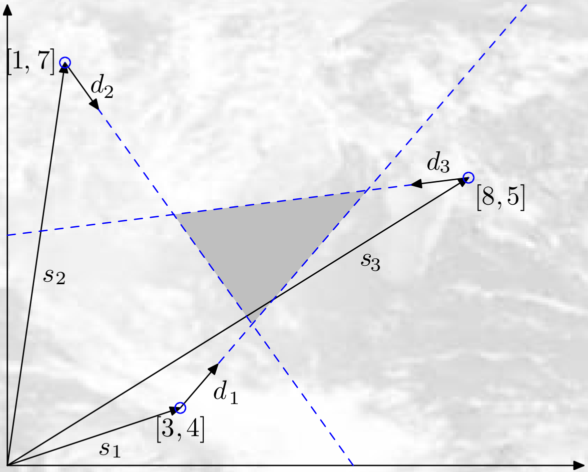

Figure 4: The gray triangle encompassed by the lines of sight of the three sensors.

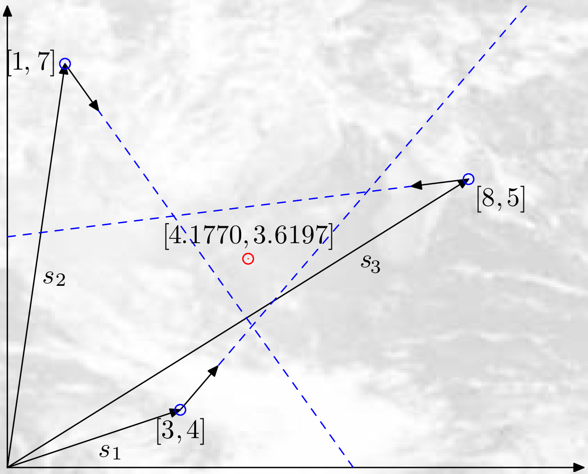

Figure 5: Our solution fits inside the gray triangle. Yeah!!

- We can apply our vectors to the equation remembering that $\mathbf{P} = \sum_{i=1}^{N} I_2 - \mathbf{d}_i \mathbf{d}_i^T$ and $\mathbf{u} = \sum_{i=1}^{N} \mathbf{s}_i - n_i \mathbf{d}_i$.

$$\sum_{i=1}^{N} [I_2 - \mathbf{d}_i \mathbf{d}_i^T] * \mathbf{r} = \sum_{i=1}^{N} \mathbf{s}_i - n_i \mathbf{d}_i$$

- We have now solved for both **P** and **u**.

$$\mathbf{P} = \begin{bmatrix} \frac{2029}{1618} & \frac{-151}{1047} \\ \frac{-151}{1047} & \frac{2825}{1618} \end{bmatrix} \approx \begin{bmatrix} 1.254 & -.1442 \\ -.1442 & 1.746 \end{bmatrix}$$

$$\mathbf{u} = \begin{bmatrix} \frac{3221}{683} \\ \frac{1538}{269} \end{bmatrix} \approx \begin{bmatrix} 4.716 \\ 5.718 \end{bmatrix}$$

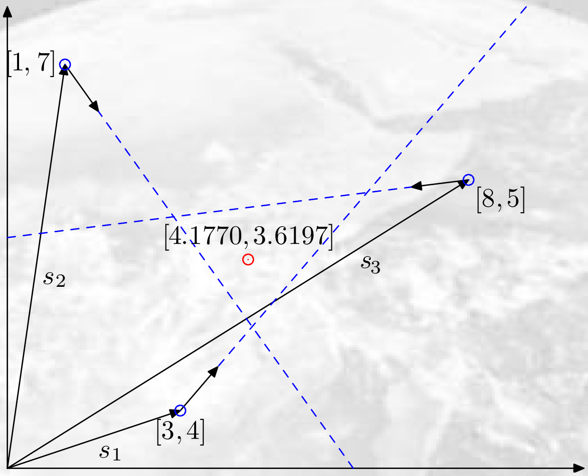

Figure 6: Yeah!!

- The vector **r** can now be found using $\mathbf{r} = \mathbf{P}^{-1} * \mathbf{u}$.

$$\mathbf{r} = \begin{bmatrix} \frac{1959}{469} \\ \frac{1104}{305} \end{bmatrix} \approx \begin{bmatrix} 4.1770 \\ 3.6197 \end{bmatrix}$$

Conclusion

- In conclusion we were successfully able to calculate the position of an object viewed from three passive sensors. Not only is GPS the trend of today, it's an accurate and reputable trend too.

References

[1] Arnold, David. For his help,support, and LaTex knowledge.

[2] Bonser, Gordon. For his help and knowledge of GPS.

[3] Draayer, Bret. "N-Site Insights". **The College Mathematics Journal**. vol. 31 no. 4 Sept 2000 250-258.

[4] Urbach, Rey. For his help and knowledge of GPS.

[5] We would like to thank Doug, Dave, Adam, and Maurissa for their help and proofreading skills.