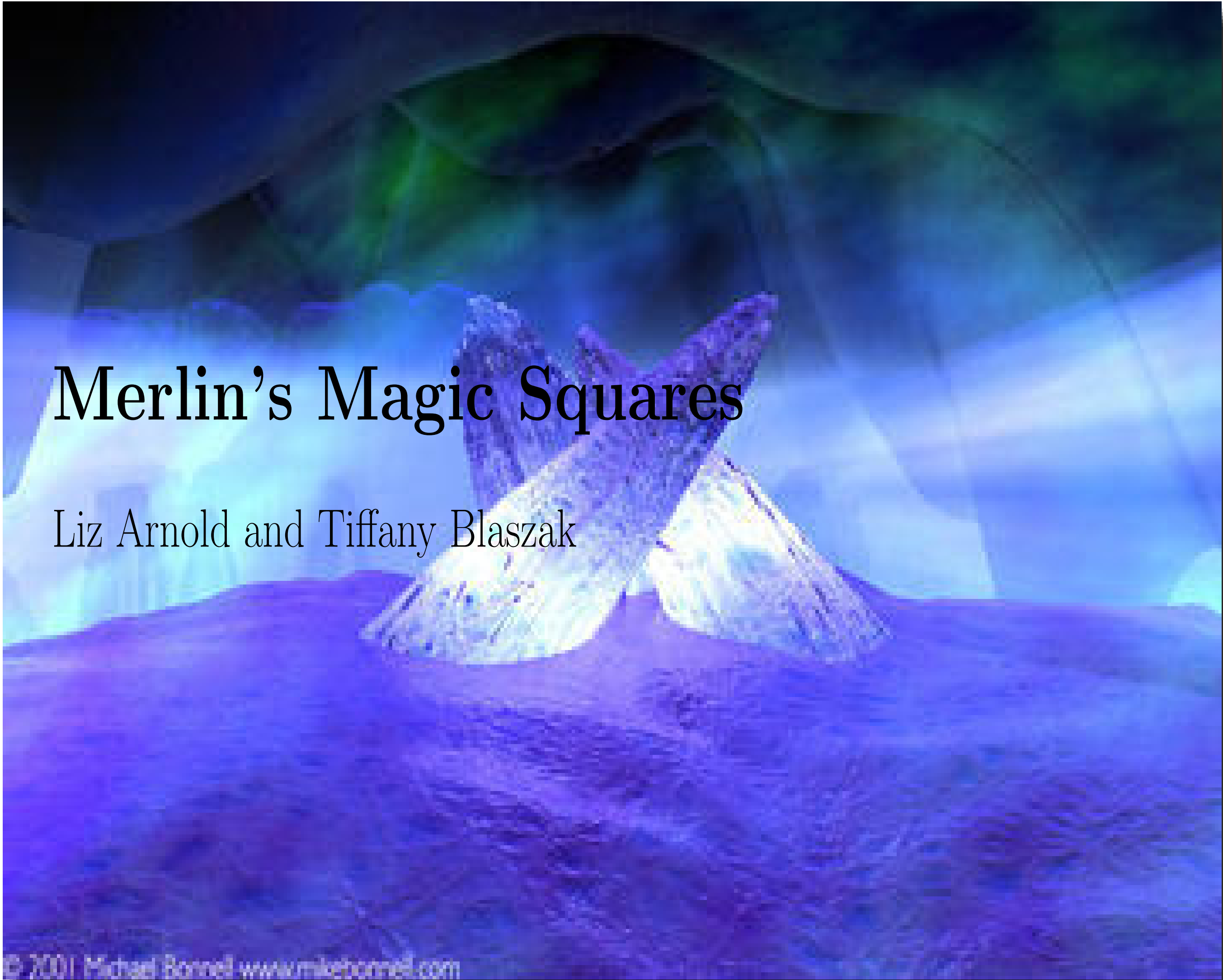

Merlin’s Magic Squares

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Introduction

Merlin's Magic Squares is a hand held electronic game made by Parker Brothers. It consists of a 3x3 array with nine boxes, each containing either a one or a zero. For our purposes we will be using the online version of the game located at http://www.cut-the-knot.com/ctk/Merlin.shtml.

Example 1

1	**0**	**0**
1	**1**	**0**
0	**0**	**1**

When a button is pressed, the button and those that surround it are changed to their opposite states in the following patterns:

0	1	0
0	0	0
0	0	1

0	1	1
0	0	1
0	0	0

0	0	1
1	1	0
0	1	0

Good Stuff

Notice that when we push the button an odd number of times, it will have the same effect as if we only pushed the button once!

1	0	0
1	1	0
0	0	1

0	1	0
0	0	0
0	0	1

1	0	0
1	1	0
0	0	1

0	1	0
0	0	0
0	0	1

And when we push the button an even number of times, it will have the same effect as if we did not push the button at all!

1	**0**	**0**
1	**1**	**0**
0	**0**	**1**

0	**1**	**0**
0	**0**	**0**
0	**0**	**1**

1	**0**	**0**
1	**1**	**0**
0	**0**	**1**

Also notice that the order in which we press the buttons does not affect the final configuration that we're shooting for.

0	1	0
0	0	0
0	0	1

0	1	1
0	0	1
0	0	0

0	0	1
1	1	0
0	1	0

1	0	1
1	1	1
0	0	0

1	1	1
0	0	0
0	1	0

0	0	1
1	1	0
0	1	0

The Goal

The goal of Merlin's Magic Squares is to reach this final configuration as shown below:

1	**1**	**1**
1	**0**	**1**
1	**1**	**1**

Using Linear Algebra

We will be using the theory of vector spaces to show a mathematical model of the game. For this game we will be working in the binary field in which the only elements are 1 and 0, and where the field operations are addition and multiplication in modulo 2 as shown below:

Addition

+	0	1
0	0	1
1	1	0

Multiplication

×	0	1
0	0	0
1	0	1

For this exercise, we can represent the configuration of our array as a column vector with nine components. It will be helpful to number its boxes and representative vector as follows:

1	**2**	**3**
4	**5**	**6**
7	**8**	**9**

$$\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{pmatrix}$$

Here we're calling our initial configuration $\mathbf{v}_p$ and our winning configuration $\mathbf{v}_w$.

$$\mathbf{v}_p = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \mathbf{v}_w = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

The game is designed such that for every possible button strike, there is a representative vector $\mathbf{u}_i$ that is added to the original vector $\mathbf{v}_p$ to produce a new vector $\mathbf{v}_{p'}$.

$$\mathbf{v}_p + \mathbf{u}_i = \mathbf{v}_{p'}$$

Example 2 *Pressing button 1 adds* $\mathbf{u}_1$ *to our initial vector as follows:*

$$\mathbf{v}_p + \mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{v}_p'$$

Note that ones appear in $\mathbf{u}_1$ corresponding to the boxes whose states are changed while zeros represent no change.

All possible button strikes then, can be represented by 9 different u_i's.

$$\mathbf{u}_1 = \begin{pmatrix}1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0\end{pmatrix}^T$$

$$\mathbf{u}_2 = \begin{pmatrix}1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}^T$$

$$\mathbf{u}_3 = \begin{pmatrix}0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0\end{pmatrix}^T$$

$$\mathbf{u}_4 = \begin{pmatrix}1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0\end{pmatrix}^T$$

$$\mathbf{u}_5 = \begin{pmatrix}0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0\end{pmatrix}^T$$

$$\mathbf{u}_6 = \begin{pmatrix}0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1\end{pmatrix}^T$$

$$\mathbf{u}_7 = \begin{pmatrix}0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0\end{pmatrix}^T$$

$$\mathbf{u}_8 = \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1\end{pmatrix}^T$$

$$\mathbf{u}_9 = \begin{pmatrix}0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1\end{pmatrix}^T$$

With this information we can now express the ability to reach the winning configuration, $\mathbf{v}_w$, by our ability to write it as a linear combination of $\mathbf{v}_p$ and our $\mathbf{u}_i$'s.

$$\mathbf{v}_w = \mathbf{v}_p + s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + ... + s_9\mathbf{u}_9$$

Our only unknowns at this point are our scalars s_i where

$$s_i = \begin{cases} 1, & \text{if } \#i \text{ is pressed,} \\ 0, & \text{otherwise.} \end{cases}$$

Now we rearrange our terms.

$$\mathbf{v}_w - \mathbf{v}_p = s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + ... + s_9\mathbf{u}_9$$

Since we are performing addition in modulo 2,

$$\mathbf{v}_w - \mathbf{v}_p = \mathbf{v}_w + \mathbf{v}_p$$

So,

$$\mathbf{v}_w + \mathbf{v}_p = s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + ... + s_9\mathbf{u}_9$$

Now we'll place our $\mathbf{u}_i$'s into the columns of a $9x9$ matrix A.

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Since our scalars multiply the columns of A, we can represent this multiplication in terms of

$$Ax = b$$

or for our purposes,

$$A\mathbf{s} = \mathbf{v}_w + \mathbf{v}_p$$

The sequence of keystrokes needed to win the game is found in $\mathbf{s}$, which is the solution to the system $A\mathbf{s} = \mathbf{v}_w + \mathbf{v}_p$, which we can rewrite to find s.

$$\begin{aligned} A\mathbf{s} &= \mathbf{v}_w + \mathbf{v}_p \\ A^{-1}\mathbf{s} &= A^{-1}(\mathbf{v}_w + \mathbf{v}_p) \\ \mathbf{s} &= A^{-1}(\mathbf{v}_w + \mathbf{v}_p) \end{aligned}$$

Now we can easily solve for s using A^{-1}.

We will need to invert A, but we must first determine whether A is invertible or not by computing the determinant.

If $\det(A) \neq 0$, then A^{-1} exists.

To find the determinant of A we will row reduce the matrix in modulo 2 and then multiply the diagonal elements:

Adding $row1$ to $row2$ gives us

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Switching $row2$ with $row3$ produces a pivot in $row2 : column\ 2$ of A.

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Continuing in this manner, we obtain an upper triangular matrix U.

$$U = \begin{pmatrix} \mathbf{1} & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix}$$

Multiplying the diagonal elements, you can see that the determinant is 1, so the matrix is indeed invertible.

Recall learning in class the equation:

$$A^{-1} = \frac{1}{|A|} C^T$$

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Here, we compute C using the Matlab's command C=mod(cofactor(A),2) which finds the cofactor matrix operating in modulo 2.

Since $|A| = 1$, we know that $A^{-1} = C^T$ so...

$$A^{-1} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Solving for s will show us the buttons we need to press to win the game. Recall:

$$A^{-1}(\mathbf{v}_w + \mathbf{v}_p) = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \mathbf{s}$$

The ones in s tell us that pressing buttons 5,6,7, and 9 will give us our winning configuration.

Conclusion

Here we see that even the most elementary linear algebra skills can be useful for real world problems.