1	0	0
1	1	0
0	0	1

Merlin's Magic Squares

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Abstract

The purpose of this paper is to solve the game Merlin's Magic Squares by using our knowledge of Linear Algebra.

Introduction

Merlin's Magic Squares is a hand held electronic game made by Parker Brothers. It consists of a 3x3 array with nine boxes, each containing either a one or a zero. For our purposes we will be using the online version of the game located at http://www.cut-the-knot.com/ctk/Merlin.shtml. When you start the game, the player is given an initial configuration. The player must then turn this initial configuration into the final configuration. When the player reaches this final configuration, the player has won the game. Figure 1 shows what an initial configuration could look like at the start of the game.

In this game, when the player presses any one of the nine buttons, a certain subset of the buttons is altered according to the rules of the game. Figures 2(a), 2(b) and 2(c) show what happens when you press a corner button, a side button and the middle

1	0	0
1	1	0
0	0	1

Figure 1: Initial Configuration.

0	1	0
0	0	0
0	0	1

(a) Corner.

0	1	1
0	0	1
0	0	0

(b) Side.

0	0	1
1	1	0
0	1	0

(c) Middle.

Figure 2: Effect of pressing a button.

1	0	0
1	1	0
0	0	1

1	**2**	**3**
4	**5**	**6**
7	**8**	**9**

Figure 3:

button respectively. This same pattern occurs for every corner button and side button that the player presses.

The nine buttons on Merlin's Magic Squares are numbered one through nine as shown in Figure 3.

Table 1 describes the effects of pressing each button.

Good Stuff

Notice that when the player pushes the same button an odd number of times, it will have the same effect as if we only pushed the button once! Here, we will show what happens to the initial configuration when the same button is pushed once, twice and three times. Notice how the effect of pushing the button once and three times produces the same configuration. Figures 4(b) and 4(d) show the effects of hitting the button once and three times respectively. Since pushing a button an odd number of times will have the same affect as if we only pushed the button once, Figure 4(b) and 4(d) will

Pressing button		alters the state of buttons
	1,	1, 2, 4 and 5
	2,	1, 2, and 3
	3,	2, 3, 5 and 6
	4,	1, 4 and 7
	5,	2, 4, 5, 6 and 8
	6,	3, 6 and 9
	7,	4, 5, 7 and 8
	8,	7, 8 and 9
	9	5, 6, 8 and 9

Table 1:

have the same configuration.

Notice in Figure 5(c) that the pressing a button twice produces the same configuration as not pressing it at all.

Figures 6 and 7 demonstrate that the order in which the player presses the buttons does not affect the final configuration.

The Goal

The goal of Merlin's Magic Squares is to reach this final configuration as shown in Figure 8.

1	0	0
1	1	0
0	0	1

1	0	0
1	1	0
0	0	1

1	0	0
1	1	0
0	0	1

(a) Initial Configuration.

0	1	0
0	0	0
0	0	1

(b) First press.

1	0	0
1	1	0
0	0	1

(c) Second press.

0	1	0
0	0	0
0	0	1

(d) Third press.

Figure 4: Effect of pressing the button in the upper left hand corner three times.

1	0	0
1	1	0
0	0	1

(a) Initial Configuration.

0	1	0
0	0	0
0	0	1

(b) First press.

1	0	0
1	1	0
0	0	1

(c) Second press.

Figure 5: Effect of pressing the button in the upper left hand corner twice.

0	1	0
0	0	0
0	0	1

(a) Button 1.

0	1	1
0	0	1
0	0	0

(b) Button 6.

0	0	1
1	1	0
0	1	0

(c) Button 5.

Figure 6: The effect of pressing buttons 1 then 6 then 5.

1	0	0
1	1	0
0	0	1

1	0	1
1	1	1
0	0	0

(a) Button 6.

1	1	1
0	0	0
0	1	0

(b) Button 5.

0	0	1
1	1	0
0	1	0

(c) Button 1.

Figure 7: The effect of pressing the same buttons in a different order (6, 5, 1).

1	1	1
1	0	1
1	1	1

Figure 8: Winning Configuration.

Using Linear Algebra

We will be using the theory of vector spaces to show a mathematical model of the game. For this game we will be working in the binary field in which the only elements are 1 and 0, and where the field operations are addition and multiplication in modulo 2 shown in the following table:

Addition

$+$	0	1
0	0	1
1	1	0

Multiplication

$\times$	0	1
0	0	0
1	0	1

For this exercise, we can represent the configuration of our array as a column vector with nine components. It will be helpful to number its boxes and representative vector in the manner shown below.

1	**2**	**3**
4	**5**	**6**
7	**8**	**9**

$$\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{pmatrix}$$

We'll call our initial configuration $\mathbf{v}_p$ and our winning configuration $\mathbf{v}_w$.

1	0	0
1	1	0
0	0	1

$$\mathbf{v}_p = \begin{pmatrix}1\\0\\0\\1\\1\\0\\0\\0\\1\end{pmatrix} \quad \text{and} \quad \mathbf{v}_w = \begin{pmatrix}1\\1\\1\\1\\0\\1\\1\\1\\1\end{pmatrix}$$

The game is designed such that for every possible button strike, there is a representative vector $\mathbf{u}_i$ that is added to the original vector $\mathbf{v}_p$ to produce a new vector $\mathbf{v}_{p'}$.

$$\mathbf{v}_p + \mathbf{u}_i = \mathbf{v}_{p'}$$

Example 1 *Pressing button 1 adds* $\mathbf{u}_1$ *to our initial vector as follows:*

$$\mathbf{v}_p + \mathbf{u}_1 = \begin{pmatrix}1\\0\\0\\1\\1\\0\\0\\0\\1\end{pmatrix} + \begin{pmatrix}1\\1\\0\\1\\1\\0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\\0\\0\\0\\0\\1\end{pmatrix} = \mathbf{v}'_p$$

Note that ones appear in $\mathbf{u}_1$ corresponding to the boxes whose states are changed while zeros represent no change.

1	**0**	**0**
1	**1**	**0**
0	**0**	**1**

1	0	0
1	1	0
0	0	1

All possible button strikes then, can be represented by 9 different u_i's.

$$\mathbf{u}_1 = \begin{pmatrix}1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0\end{pmatrix}^T$$
$$\mathbf{u}_2 = \begin{pmatrix}1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}^T$$
$$\mathbf{u}_3 = \begin{pmatrix}0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0\end{pmatrix}^T$$
$$\mathbf{u}_4 = \begin{pmatrix}1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0\end{pmatrix}^T$$
$$\mathbf{u}_5 = \begin{pmatrix}0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0\end{pmatrix}^T$$
$$\mathbf{u}_6 = \begin{pmatrix}0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1\end{pmatrix}^T$$
$$\mathbf{u}_7 = \begin{pmatrix}0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0\end{pmatrix}^T$$
$$\mathbf{u}_8 = \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1\end{pmatrix}^T$$
$$\mathbf{u}_9 = \begin{pmatrix}0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1\end{pmatrix}^T$$

With this information we can now express the ability to reach the winning configuration, $\mathbf{v}_w$, by our ability to write it as a linear combination of $\mathbf{v}_p$ and our $\mathbf{u}_i$'s.

$$\mathbf{v}_w = \mathbf{v}_p + s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_9\mathbf{u}_9$$

Our only unknowns at this point are our scalars s_i where

$$s_i = \begin{cases} 1, & \text{if button } \#i \text{ is pressed,} \\ 0, & \text{otherwise.} \end{cases}$$

Now we rearrange our terms.

$$\mathbf{v}_w - \mathbf{v}_p = s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_9\mathbf{u}_9$$

1	0	0
1	1	0
0	0	1

Since we are performing addition in modulo 2,

$$\mathbf{v}_w - \mathbf{v}_p = \mathbf{v}_w + \mathbf{v}_p$$

So,

$$\mathbf{v}_w + \mathbf{v}_p = s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_9\mathbf{u}_9$$

Now we'll place our $\mathbf{u}_i$'s into the columns of a 9×9 matrix A.

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Since our scalars multiply the columns of A, we can represent this multiplication as

$$Ax = b,$$

or for our purposes,

$$A\mathbf{s} = \mathbf{v}_w + \mathbf{v}_p.$$

The sequence of keystrokes needed to win the game is found in $\mathbf{s}$, which is the solution to the system $A\mathbf{s} = \mathbf{v}_w + \mathbf{v}_p$, which we can rewrite to find s.

$$\begin{aligned} A\mathbf{s} &= \mathbf{v}_w + \mathbf{v}_p \\ A^{-1}\mathbf{s} &= A^{-1}(\mathbf{v}_w + \mathbf{v}_p) \\ \mathbf{s} &= A^{-1}(\mathbf{v}_w + \mathbf{v}_p) \end{aligned}$$

Now we can easily solve for s using A^{-1}.

We will need to invert A, but we must first determine whether A is invertible or not by computing the determinant.

$$\text{If } \det(A) \neq 0, \text{ then } A^{-1} \text{ exists.}$$

To find the determinant of A we will row reduce the matrix in modulo 2 and then multiply the diagonal elements:

Adding $row1$ to $row2$ gives us

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Switching $row2$ with $row3$ produces a pivot in $row2 : column\ 2$ of A.

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Continuing in this manner, we obtain an upper triangular matrix U.

$$U = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Multiplying the diagonal elements, you can see that the determinant is 1, so the matrix is indeed invertible.

A formula for calculating A^{-1} is:

$$A^{-1} = \frac{1}{|A|} C^T$$

where C is the cofactor matrix of A.

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Here, we compute C using the Matlab's command C=mod(cofactor(A),2), which finds the cofactor matrix operating in modulo 2.

Since $|A| = 1$, we know that $A^{-1} = C^T$ so,

$$A^{-1} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Solving for $\mathbf{s}$ will show us the buttons we need to press to win the game. Recall that

$$A^{-1}(\mathbf{v}_w + \mathbf{v}_p) = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \mathbf{s}.$$

The ones in $\mathbf{s}$ tell us that pressing buttons 5, 6, 7, and 9 will give us our winning configuration. The winning play is shown in Figure 9.

1	0	0
1	1	0
0	0	1

(a)

1	1	0
0	0	1
0	1	1

(b)

1	1	1
0	0	0
0	1	0

(c)

1	1	1
1	1	0
1	0	0

(d)

1	1	1
1	0	1
1	1	1

(e)

Figure 9: Pressing buttons 5, 6, 7, and 9 wins the game.

Conclusion

Now we have a surefire method for winning Merlin's Magic Squares given any initial configuration. In fact, it is possible to achieve any desired configuration by simply replacing our original goal configuration with some new configuration of our choice, then following the same steps. This application, though not particularly technical or complicated, demonstrates that we can apply linear algebra to real situations. Our simple example shows us a fun way to exercise our linear algebra skills.

References

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