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Image Compression Using the Haar Wavelet Transform

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Purpose

Digital images require large amounts of memory to store and, when retrieved from the internet, can take a considerable amount of time to download. The Haar wavelet transform provides a method of compressing image data so that it takes up less memory. Today I will discuss the implementation of the Haar wavelet transform and some of its applications.

Background

The files that comprise these images can be quite large and can quickly take up precious memory space on the computer's hard drive. A gray scale image that is 256 x 256 pixels has $65,536$ elements to store, and a typical 640 x 480 color image has nearly a million! The size of these files can also make downloading from the internet a lengthy process. The Haar wavelet transform provides a means by which we can compress the image so that it takes up much less storage space, and therefore transmits faster electronically and in progressive levels of detail.

How Images are Stored

Before we can understand how to manipulate an image, it is important to understand exactly how the computer stores the image. A 256 x 256 pixel gray scale image is stored as a 256 x 256 matrix, with each element of the matrix being a number ranging from zero (for black) to some positive whole number (for white). A 256 x 256 color image is stored as three 256 x 256 matrices (One each for the colors red, green, and blue). We can use this matrix, and some linear algebra to maximize compression while maintaining a suitable level of detail.

How the Wavelet Transform Works

- The Haar wavelet uses a method of manipulating matrices called averaging and differencing.
- If $\mathbf{y}$ is a row of an image matrix, when it is averaged and differenced we get the following results:

$\mathbf{y}$	448	768	704	640	1280	1408	1600	1600
$\mathbf{y}_1$	608	672	1344	1600	-160	32	-64	0
$\mathbf{y}_2$	640	1472	-32	-128	-160	32	-64	0
$\mathbf{y}_3$	1056	-416	-32	-128	-160	32	-64	0

- Blue = *Approximation Coefficients*
- Red = *Detail Coefficients*

Using Matrix Multiplication

But,

$$\mathbf{y_1} = \mathbf{y}\, A_1$$

Where A_1 is the matrix

$$A_1 = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & -1/2 \end{pmatrix}$$

And,

$$\mathbf{y_2} = \mathbf{y_1}\, A_2$$

Where A_2 is the matrix

$$A_2 = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & -1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

And,

$$\mathbf{y_3} = \mathbf{y_2}\, A_3$$

Where A_3 is the matrix

$$A_3 = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The Power of Linear Algebra

- This set of operations can all be done in one step:

$$\mathbf{y_3} = \mathbf{y}\, W$$

- Where W is is the transformation matrix:

$$W = A_1\, A_2\, A_3$$

- Also note that since each column of the A_i matrices that comprise W is orthogonal to every other, the matrices are invertible. Thus:

$$W^{-1} = A_3^{-1}\, A_2^{-1}\, A_1^{-1}$$

- This means that we can get our original data back using the equation:

$$\mathbf{y} = \mathbf{y_3} W^{-1}$$

- In general we can say that $Q = I W$, where Q is the row transformed matrix and I is the original image matrix. But, as I stated before, the Haar wavelet transformation does these transformations to each column of the image matrix, *and then repeats them on each column* of the matrix. This is done by multiplying I on the left by the transpose of W. This gives us our final equation for the *row-and-column* transformed matrix T:

$$T = W^T I W$$

- It also follows from this that we can get back to our original image matrix I using the following equation:

$$I = (W^T)^{-1} T W^{-1}$$

An Example

Suppose we start with an 8 x 8 image represented by matrix P:

$$P = \begin{pmatrix} 576 & 704 & 1152 & 1280 & 1344 & 1472 & 1536 & 1536 \\ 704 & 640 & 1156 & 1088 & 1344 & 1408 & 1536 & 1600 \\ 768 & 832 & 1216 & 1472 & 1472 & 1536 & 1600 & 1600 \\ 832 & 832 & 960 & 1344 & 1536 & 1536 & 1600 & 1536 \\ 832 & 832 & 960 & 1216 & 1536 & 1600 & 1536 & 1536 \\ 960 & 896 & 896 & 1088 & 1600 & 1600 & 1600 & 1536 \\ 768 & 768 & 832 & 832 & 1280 & 1472 & 1600 & 1600 \\ \mathbf{448} & \mathbf{768} & \mathbf{704} & \mathbf{640} & \mathbf{1280} & \mathbf{1408} & \mathbf{1600} & \mathbf{1600} \end{pmatrix}$$

Notice that the last row is our vector $\mathbf{y}$.

First we want to average and difference the rows of matrix P. In order to get the row averaged matrix Q we simply multiply P on the right by matrix W, our transformation matrix. This yields:

$$Q = \begin{pmatrix} 1200 & -272 & -288 & -64 & -64 & -64 & -64 & 0 \\ 1185 & -288 & -225 & -96 & 32 & 34 & -32 & -32 \\ 1312 & -240 & -272 & -48 & -32 & -128 & -32 & 0 \\ 1272 & -280 & -160 & -16 & 0 & -192 & 0 & 32 \\ 1256 & -296 & -128 & 16 & 0 & -128 & -32 & 0 \\ 1272 & -312 & -32 & 16 & 32 & -96 & 0 & 32 \\ 1144 & -344 & -32 & -112 & 0 & 0 & -96 & 0 \\ 1056 & \mathbf{-416} & \mathbf{-32} & \mathbf{-128} & \mathbf{-160} & \mathbf{32} & \mathbf{-64} & \mathbf{0} \end{pmatrix}$$

Notice that the last row in the matrix is identical to $\mathbf{y_3}$.

We can now average and difference the columns of Q to get our row-and-column transformed matrix T by multiplying Q on the left by W^T. This yields:

$$T = \begin{pmatrix} 1212 & -306 & -146 & -54 & -24 & -68 & -40 & 4 \\ 30 & 36 & -90 & -2 & 8 & -20 & 8 & -4 \\ -50 & -10 & -20 & -24 & 0 & 72 & -16 & -16 \\ 82 & 38 & -24 & 68 & 48 & -64 & 32 & 8 \\ 8 & 8 & -32 & 16 & -48 & -48 & -16 & 16 \\ 20 & 20 & -56 & -16 & -16 & 32 & -16 & -16 \\ -8 & 8 & -48 & 0 & -16 & -16 & -16 & -16 \\ 44 & 36 & 0 & 8 & 80 & -16 & -16 & 0 \end{pmatrix}$$

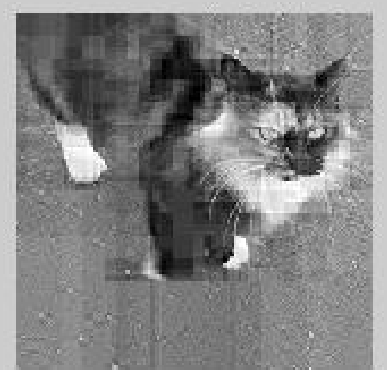
Now matrix T is ready to be compressed.

Even More Power

Another way to make these transformation matrices more powerful is to *normalize* our transformation matrix W by dividing each column of W by its length. The result is that W is *orthogonal*, meaning that each column of the matrix is length one, and orthogonal to each other column of the matrix. This is called a *normalized wavelet transform*. Having an orthogonal transformation matrix has two major benefits:

1. First, because the inverse of an orthogonal matrix is equal to its transpose, this greatly increases the speed at which we can perform the calculations that reconstitute our image matrix.

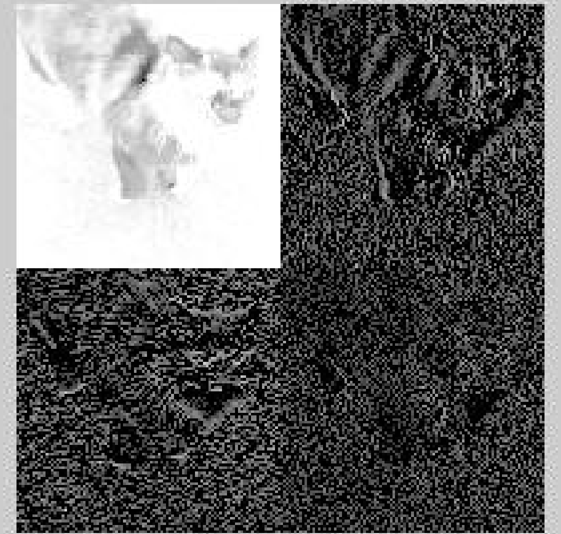
2. Second, as you can see from the following figure (compressed to $5.5 : 1$), normalized transformations tend be closer to the original image.



Approximation and Detail Coefficients

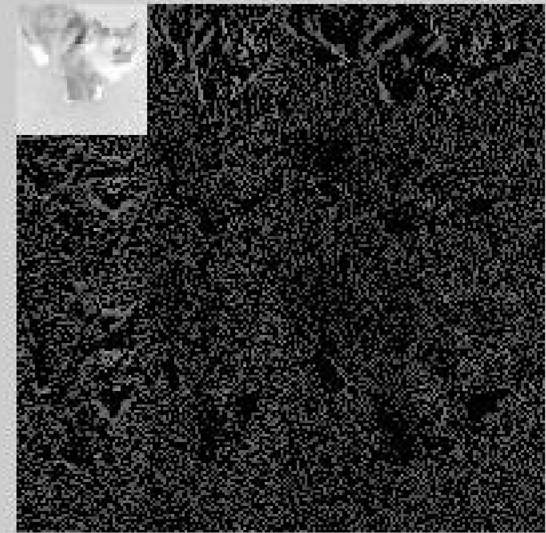
If we were to average and difference each row and column of matrix P only once ($T_1 = A_1^T \, P \, A_1$), we would essentially divide our 8 x 8 matrix into four 4 x 4 sub-matrices:

$$T_1 = \begin{pmatrix} 656 & 1169 & 1392 & 1552 & -16 & -15 & -48 & -16 \\ 816 & 1248 & 1520 & 1584 & -16 & -160 & -16 & 16 \\ 880 & 1040 & 1584 & 1552 & 16 & -112 & -16 & 16 \\ 688 & 752 & 1360 & 1600 & -80 & 16 & -80 & 0 \\ -16 & 47 & 16 & -16 & -48 & -49 & -16 & 16 \\ -16 & 96 & -16 & 16 & -16 & 32 & -16 & -16 \\ -48 & 48 & -16 & -16 & -16 & -16 & -16 & -16 \\ 80 & 80 & 16 & 0 & 80 & -16 & -16 & 0 \end{pmatrix}$$



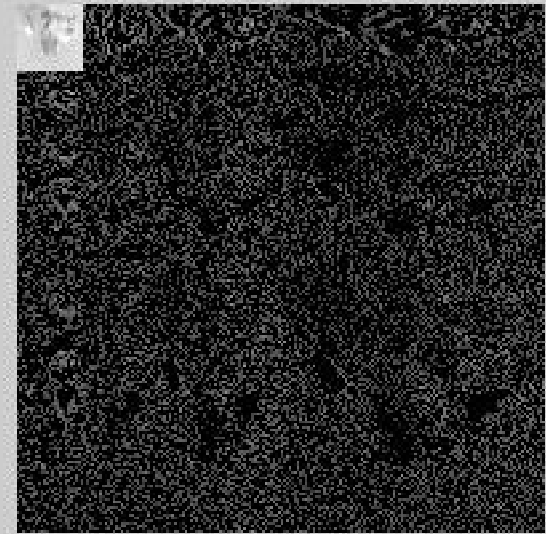
If we take P and perform two rounds of averaging and differencing on it ($T_2 = A_2^T \, A_1^T \, P \, A_1 \, A_2$), we end up with the following matrix:

$$T_2 = \begin{pmatrix} 972 & 1512 & -236 & -56 & -16 & -88 & -32 & 0 \\ 840 & 1524 & -56 & -52 & -32 & -48 & -48 & 8 \\ -60 & -40 & -20 & -24 & 0 & 73 & -16 & -16 \\ 120 & 44 & -24 & 68 & 48 & -64 & 32 & 8 \\ 16 & 0 & -32 & 16 & -48 & -49 & -16 & 16 \\ 40 & 0 & -56 & -16 & -16 & 32 & -16 & -16 \\ 0 & -16 & -48 & 0 & -16 & -16 & -16 & -16 \\ 80 & 8 & 0 & 8 & 80 & -16 & -16 & 0 \end{pmatrix}$$



If we take P and perform three rounds of averaging and differencing on it ($T = A_3^T A_2^T A_1^T P A_1 A_2 A_3 = W^T P W$), we end up with the same matrix T we saw in the earlier example:

$$T = \begin{pmatrix} 1212 & -306 & -146 & -54 & -24 & -68 & -40 & 4 \\ 30 & 36 & -90 & -2 & 8 & -20 & 8 & -4 \\ -50 & -10 & -20 & -24 & 0 & 72 & -16 & -16 \\ 82 & 38 & -24 & 68 & 48 & -64 & 32 & 8 \\ 8 & 8 & -32 & 16 & -48 & -48 & -16 & 16 \\ 20 & 20 & -56 & -16 & -16 & 32 & -16 & -16 \\ -8 & 8 & -48 & 0 & -16 & -16 & -16 & -16 \\ 44 & 36 & 0 & 8 & 80 & -16 & -16 & 0 \end{pmatrix}$$



Wavelet Compression

Image compression using the Haar wavelet transform can be summed up in a few simple steps.

1. Convert the image into a matrix format(I).
2. Calculate the row-and-column transformed matrix (T) using $T = W^T\, I\, W$. The transformed matrix should be relatively sparse.
3. Select a threshold value ϵ, and replace any element of T less than ϵ with a zero. This will result in a sparse matrix denoted as S.
4. To get our reconstructed matrix R from matrix S, we use an equation similar to $I = (W^T)^{-1}\, T\, W^{-1}$, but, because the inverse of an orthogonal matrix is equal to its transpose, we modify the equation as follows:

$$R = WSW^{-1}$$

- If $\epsilon = 0$, then $S = T$ and therefore $R = I$. This is called *lossless compression*.
- If $\epsilon > 0$, then elements of T are reset to zero, and some original data is lost. The reconstituted image will contain distortions. This is called *lossy compression*.
- The secret of optimal compression is to choose ϵ so that compression is maximized while distortions in the reconstituted image are minimized.
- The level of compression is measured by the *compression ratio*, which is the ratio of nonzero entries in the transformed matrix T to the number of nonzero entries in the compressed matrix S. According to Dr. Colm Mulcahy of Spelman University, a compression ratio of $10:1$ or greater means that S is sparse enough to have a significant savings in terms of storage and transmission time.

Examples

Let matrix I be the 8 by 8 matrix that makes up the upper left hand corner of the original image:

$$I = \begin{pmatrix} 100 & 103 & 99 & 97 & 93 & 94 & 78 & 73 \\ 102 & 97 & 100 & 111 & 113 & 104 & 96 & 82 \\ 99 & 109 & 104 & 95 & 93 & 92 & 88 & 76 \\ 114 & 104 & 99 & 102 & 93 & 82 & 74 & 74 \\ 96 & 91 & 91 & 87 & 79 & 78 & 77 & 76 \\ 90 & 88 & 83 & 78 & 77 & 74 & 76 & 76 \\ 92 & 81 & 73 & 72 & 69 & 65 & 66 & 62 \\ 75 & 70 & 69 & 65 & 60 & 55 & 61 & 65 \end{pmatrix}$$

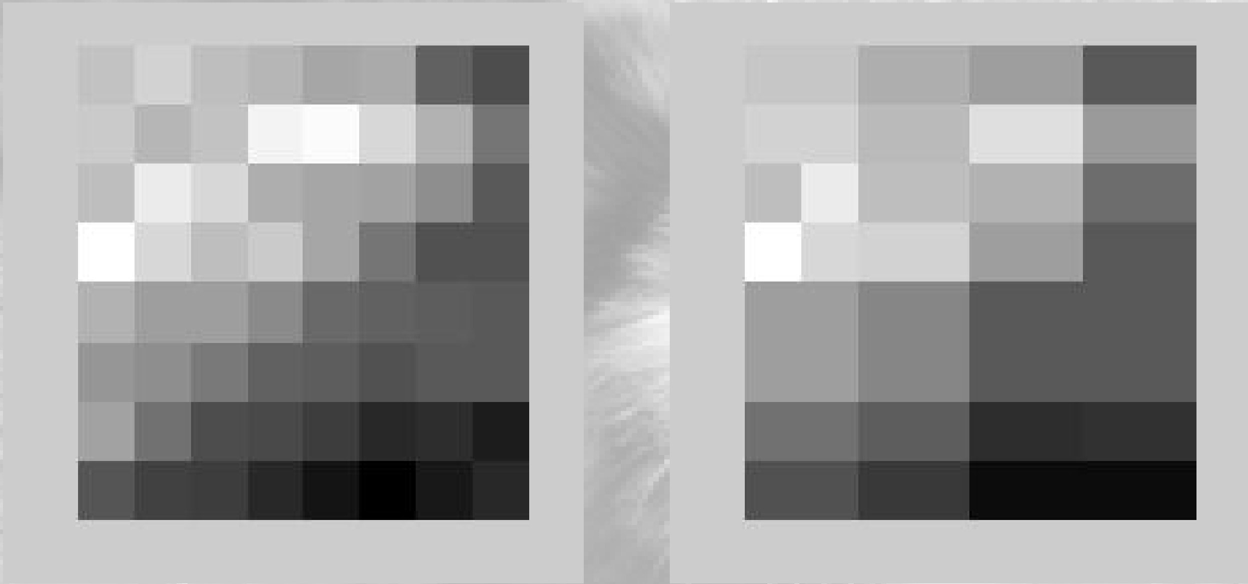
When we perform the normalized transformation on I, and choose ϵ such that we get a compression ratio of approximately $5 : 1$, we get the following compressed matrix:

$$S = \begin{pmatrix} 255 & 52 & 15 & 21 & 0 & 0 & 0 & 0 \\ 78 & 0 & 0 & 22 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 38 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 11 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

When we restore the matrix using $R = W\,S\,W^{-1}$ we end up with the following reconstructed matrix:

$$R = \begin{pmatrix} 100 & 100 & 95 & 95 & 92 & 92 & 76 & 76 \\ 103 & 103 & 98 & 98 & 106 & 106 & 90 & 90 \\ 99 & 109 & 99 & 99 & 96 & 96 & 81 & 81 \\ 114 & 104 & 104 & 104 & 91 & 91 & 76 & 76 \\ 91 & 91 & 86 & 86 & 76 & 76 & 76 & 76 \\ 91 & 91 & 86 & 86 & 76 & 76 & 76 & 76 \\ 82 & 82 & 76 & 76 & 66 & 66 & 66 & 66 \\ 74 & 74 & 69 & 69 & 58 & 58 & 59 & 59 \end{pmatrix}$$

If you compare matrix R with matrix I, it is easy to see that although they are similar, there are small variations. These variations show up as distortions in the reconstructed image.



Electronic Transmission

People frequently download images from the internet, a few of which are not even pornographic. Wavelet transforms enhance electronic transfers of images in two ways:

- Because a compressed image file is significantly smaller it takes far less time to download.
- Progressive Transmission - When an image is requested, the source computer first sends the overall approximation coefficient and larger detail coefficients, and then the progressively smaller detail coefficients. As your computer receives this information it begins to reconstruct the image in progressively greater detail until the original image is fully reconstructed. This process can be interrupted at any time. Otherwise an user would only be able to see an image after the entire image file had been downloaded.

(a) Original Image (b) Rebuilt Compressed Image

Figure 1: Image with 10.3 : 1 compression ratio.

(a) Original Image (b) Rebuilt Compressed Image

Figure 2: Image with 10.2 : 1 compression ratio.

References

[1] Ames, Rebecca *For instruction in Adobe Photoshop*

[2] Arnold, Dave *College of the Redwoods*

[3] Huang, LiHong *For his help in constructing the Matlab M-Files*

[4] Mulcahy, Colm *Image Compression Using the Haar Wavelet Transform*

[5] Strang, Gilbert *Introduction to Linear Algebra*