Circulant Matrices and Polynomials

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$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 8 & 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 7 & 8 & 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 & 8 & 1 & 2 & 3 \\ 3 & 4 & 5 & 6 & 7 & 8 & 1 & 2 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1 \end{pmatrix}$$

What is a Circulant Matrix?

An $n \times n$ *circulant matrix* is formed by starting with a vector with n components. This vector becomes the first row of the matrix. Subsequent rows shift the elements of the previous row to the right.

Examples:

$$C = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix}$$

Generating a Circulant

First we define the generator matrix W:

- For the 3×3 generator matrix, W is defined as:

$$W = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

- We also need to know that:

$$W^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

- To generate a 3×3 circulant, let:

$$q(t) = a + bt + ct^2$$

- Now we evaluate:

$$\begin{aligned} C &= q(W) \\ &= aI + bW + cW^2 \\ &= a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & b \\ b & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & c \\ c & 0 & 0 \\ 0 & c & 0 \end{pmatrix} \\ &= \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} \end{aligned}$$

A 4×4 Example

Of course, the generating matrices are now 4×4.

$$W = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad W^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad W^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Generating the Circulant

Now suppose we want to generate a 4×4 circulant with $[a, b, c, d]$ for its first row. Evaluate

$$q(t) = a + bt + ct^2 + dt^3$$

at W.

$$\begin{aligned} C &= q(W) \\ &= aI + bW + cW^2 + dW^3 \\ &= \begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix} \end{aligned}$$

"What concerns us most about circulant matrices is the simple computation of their eigenvalues using nth roots of unity."

The nth Roots of Unity

- When we say "the nth roots of unity", we mean that we want the solutions to:

$$z^n = 1$$

- And we know that there must be exactly n solutions.

Square Roots of Unity

- To find the square roots of unity, we solve:

$$z^2 = 1$$

- We know that our two solutions are $z = \pm 1$.
- However, for exponents greater than 2, we will find that there must be complex solutions.

The Complex Plane

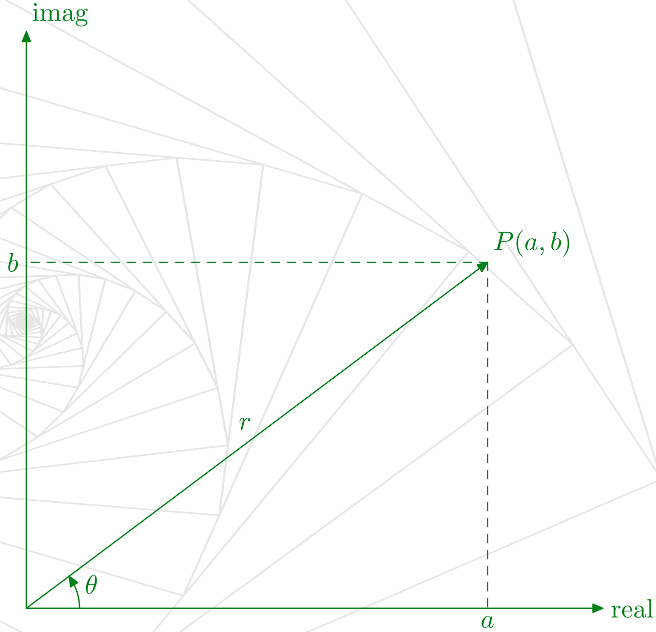
Recall that any complex number $z = a + bi$ with coordinates (a, b) can be plotted on the complex plane.

$$z = a + bi$$

$$z = r\left(\frac{a}{r} + i\frac{b}{r}\right)$$

$$z = r(\cos\theta + i\sin\theta)$$

$$z = re^{i\theta}$$



Cube Roots of Unity

When we want the cube roots of unity, we solve

$$z^3 = 1.$$

Letting $z = re^{i\theta}$,

$$(re^{i\theta})^3 = e^{i2k\pi}$$
$$r^3 e^{i3\theta} = e^{i2k\pi}$$

Thus,

$$r^3 = 1 \qquad 3\theta = 2k\pi$$
$$r = 1 \qquad \theta = \frac{2k\pi}{3}$$

Substituting in $z = re^{i\theta}$,

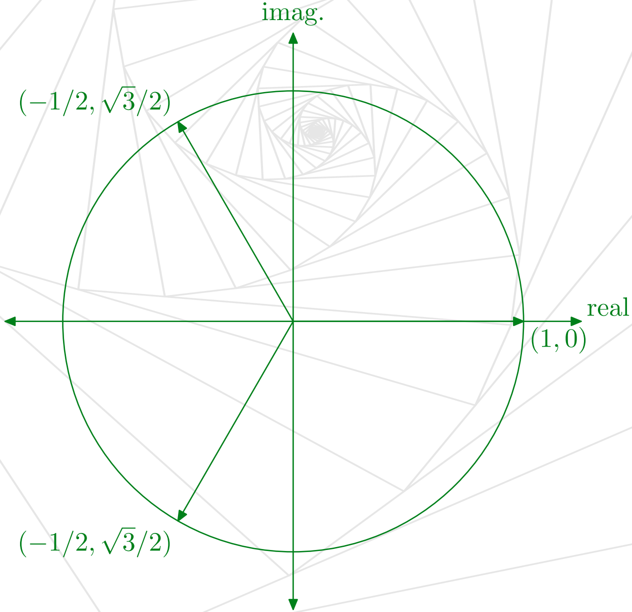
$$z = e^{i2k\pi/3}$$

Now our three roots are given by $k = 0$, 1, and 2.

$$z_0 = e^0 = 1$$

$$z_1 = e^{i2\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$z_2 = e^{i4\pi/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$



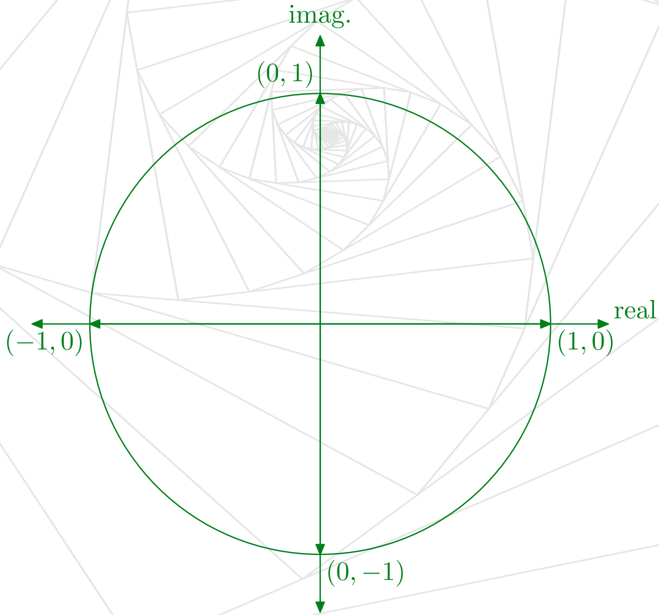
Fourth Roots of Unity

We are also interested in the fourth roots of unity, so we solve:

$$z^4 = 1$$

By solving for z in the same manner as before, we arrive at:

$$z = e^{ik\pi/2}$$



Eigenvalues

- We know that given a matrix A with eigenvalues λ:

$$A\mathbf{x} = \lambda\mathbf{x}$$

- The interesting implication is that:

$$q(A)\mathbf{x} = q(\lambda)\mathbf{x}$$

- That is, the number $q(\lambda)$ is an eigenvalue of the matrix $q(A)$.

Example

Let

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 3$. Now, let

$$q(t) = 1 + 2t + t^2.$$

Now we evaluate

$$\begin{aligned} q(A) &= I + 2A + A^2 \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}^2 \\ &= \begin{pmatrix} 4 & 14 \\ 0 & 16 \end{pmatrix} \end{aligned}$$

Note that the eigenvalues are $\lambda = 4$ and $\lambda = 16$.

Further, note that with $\lambda = 1$,

$$q(1) = 1 + 2(1) + (1)^2 = 4$$

and when $\lambda = 3$,

$$q(3) = 1 + 2(3) + (3)^2 = 16$$

In summary, we have shown that:

- The eigenvalues of A are $\lambda = 1$ and $\lambda = 3$.
- The eigenvalues of $q(A)$, by inspection, are $\lambda = 4$ and $\lambda = 16$.
- However, the eigenvalues of $q(A)$ are also given by $q(1)$ and $q(3)$.

Eigenvalues of a Circulant

- We want to calculate the eigenvalues of a given circulant $C = q(W)$.
- But, we now know that if λ is an eigenvalue of W, then $q(\lambda)$ is an eigenvalue of $q(W)$.
- So we want the eigenvalues of W and we solve:

$$\det(W - \lambda I) = 0$$
$$\lambda^2(-\lambda + \frac{1}{\lambda^2}) = 0$$
$$-\lambda^3 + 1 = 0$$
$$\lambda^3 = 1$$

- The eigenvalues of W are precisely the cube roots of unity.
- Likewise, the eigenvalues of the 4×4 generator W are the fourth roots of unity.

We can now say that

$$C\mathbf{x} = q(n)\mathbf{x}$$

where n is the nth roots of unity. Therefore, the values of $q(n)$ are the eigenvalues of C. But, q is defined by the first row of C.

So, we have now arrived at a simple method for calculating the eigenvalues of any $n \times n$ circulant matrix.

- Write down the polynomial q, in ascending order, defined by the first row of C.
- Evaluate q at the nth roots of unity.
- The values of $q(n)$ are now the eigenvalues of C.

Example

Consider the 4×4 circulant:

$$C = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 3 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 3 & 1 \end{pmatrix}$$

Define: $q(t) = 1 + 2t + t^2 + 3t^3$

The eigenvalues of C are now: $q(1) = 7$, $q(-1) = -3$, $q(i) = -i$, and $q(-i) = i$. Where we have evaluated q at the fourth roots of unity.

```
>> eig(C)
ans =
   7.0000
  -3.0000
   0.0000 + 1.0000i
   0.0000 - 1.0000i
```

Solving Polynomials With Circulants

Using circulants, we have a new method for solving polynomials.

- We are given a polynomial p.
- We need to find a circulant matrix whose characteristic polynomial is p.
- The eigenvalues $q(n)$ are now the roots of p.

A Quadratic Example

$$p(x) = x^2 - 2x - 3$$

Now we want to find a 2×2 circulant matrix whose characteristic polynomial is p. So, we must consider the general 2×2 circulant

$$C = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

The characteristic polynomial of C is then given by

$$\det(xI - C) = \det \begin{pmatrix} x - a & b \\ b & x - a \end{pmatrix} = x^2 - 2ax + a^2 - b^2$$

By inspection we can make the following relationships:

$$-2a = -2$$
$$a^2 - b^2 = -3$$

Solving this system we get $a = 1$ and $b = \pm 2$. For simplicity we take $b = 2$ and we have

$$C = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

The first row of C now defines q

$$q(t) = 1 + 2t$$

Finally, evaluating q at the two square roots of unity, gives the roots of p.

$$q(1) = 1 + 2(1) = 3$$
$$q(-1) = 1 + 2(-1) = -1$$

The Quadratic Formula

Now, we want to solve a general quadratic polynomial

$$p(t) = t^2 + \alpha t + \beta$$

So, again we find

$$\det(xI - C) = x^2 - 2ax + a^2 - b^2$$

Now we need to find a and b so that our above expression equals p.

$$-2a = \alpha$$
$$a^2 - b^2 = \beta$$

Thus,

$$a = -\frac{\alpha}{2} \qquad \text{and} \qquad b = \pm\sqrt{\frac{\alpha^2}{4} - \beta}$$

Substituting a and b into our circulant C, we get

$$C = \begin{pmatrix} -\frac{\alpha}{2} & \sqrt{\frac{\alpha^2}{4} - \beta} \\ \sqrt{\frac{\alpha^2}{4} - \beta} & -\frac{\alpha}{2} \end{pmatrix}$$

And by inspection we have

$$q(t) = -\frac{\alpha}{2} + t\sqrt{\frac{\alpha^2}{4} - \beta}$$

So, the roots of p are now the values of q evaluated at the two square roots of unity, 1 and -1.

$$q(1) = -\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - \beta}$$

$$q(-1) = -\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - \beta}$$

The Cubic, Quartic, and Beyond

- The same analysis extends naturally to the cubic and the quartic.
- However, the method fails for higher degree polynomials.
- In conclusion, the circulant method provides a simple, unified approach to polynomial solutions through degree four.