Arnold's Cat Map

Gabriel Peterson
Math 45 – Linear Algebra
Fall 1997

Abstract

The purpose of this paper is to discuss and explore the properties of Arnold's cat map, a chaotic mapping of the pixels of an image. By way of this, the introductory principles of the exciting new science of chaos are addressed.

Introduction: Chaos

Before undertaking a discussion of Arnold's cat map, a working understanding of chaos must first be established. We enter into this discussion already in the possession of some intuitive sense regarding chaos, some nebulous sense that it involves disorder, randomness, disorganization, entropy... Who among us has not observed that, over time, systems tend to become disorganized or chaotic. Does a bedroom not evolve from an initial state of order to disorder with indifferent cleaning? A jar containing marbles of two different colors, initially group by color, promptly distributes the marbles randomly when vigorously shaken. Indeed, the inviolable and sacrosanct **Second Law of Thermodynamics** tells as much: the entropy, the measure of disorder, of the Universe increases over time.

However, the chaos that concerns this discussion, while inextricably tied to the above, is that of the fledgling science of chaos. This science has leapt into the public consciousness in recent years with popularization in motion pictures, such as **Jurassic Park**, and books, such as James Gleick's **Chaos**: **Making a New Science**. Chaos evolved from the work of Edward Lorenz, a meteorologist at Massachusetts Institute of Technology. In 1960, equipped with a newly purchased electronic computer, he created a simulation of the weather using a simple system of equations; the machine had neither the computational power nor the memory for a more sophisticated model. Initial conditions were entered, and the weather in this electronic world would unfold, its binary denizens enduring downpours, blizzards, droughts, and other meteorological adversities. Wanting to examine a particular run in greater detail, Lorenz reenter its initial conditions. However, the data from the first run had been entered to six decimal places of accuracy, whereas the second only to three. His discovery was startling: comparing the second run to the first – which should have been virtually identical – he found that the second rapidly diverged from the first. Surely, the small error introduced by truncating the initial conditions to three decimal places could not have induced this troubling behavior. But it did. Complex systems – even a simplified complex system such as Lorenz's – are extremely sensitive to initial conditions, so-called *sensitivity dependence upon initial conditions*. Small errors rapidly propagate through such systems. Lorenz called this the Butterfly Effect: the beating of a butterfly's wings in China could cause a blizzard in Chicago.

It was not until the publishing of the paper mischievously titled "Period Three Implies Chaos" in 1975 by James Yorke and Tien-Yien Li that the word chaos was coined mathematically to give these ideas a name. Broadly defined, chaos pertains to those mathematical or physical phenomena that are apparently random yet possess underlying order. As James Gleick commented,

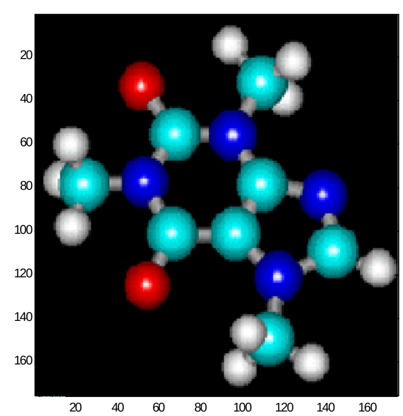
> Over the last decade, physicists, biologists, astronomers, and economists have created a new way of understanding the growth of complexity in nature. This new science, called chaos, offers a way if seeing order and pattern where formerly only the random, erratic, the unpredictable – in short, the chaotic – had been observed (Chaos and Fractals, vii).

In a now famous series of debates with Niels Bohr concerning quantum physics, Albert Einstien proclaimed "God does not play dice with the Universe," the Universe is not governed by chance. However, chaos implies, in the words of Joseph Ford of the Georgia Institute of Technology,

> God does play dice with the Universe. But they are loaded dice. And the main objective of physics now is to find out by what rules were they loaded and how can we use them for our own end (Chaos, 314).

Introduction: Images

As matter is composed of discrete units called atoms (which are themselves composed of discrete units), so too images are composed of discrete units called pixels. A pixel is a small square representing some color value, which when taken together form the mosaic that is the image. The image is a $m \times n$ matrix, where m represents the number of rows of pixels and n the number of columns of pixels, with each entry in the matrix being a numeric value that represents a given color. For example, consider the 175 x 175 image of a caffeine molecule below.



Let the image be the matrix X, and we can examine selected entries in X.

$$
X = \begin{bmatrix}
217 & 217 & 217 & 217 & ... & 217 & 217 & 217 & 217 \\
251 & 251 & 251 & 251 & ... & 251 & 251 & 251 & 251 \\
251 & 251 & 251 & 251 & ... & 251 & 251 & 251 & 251 \\
251 & 251 & 251 & 251 & ... & 251 & 251 & 251 & 251 \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
251 & 251 & 251 & 251 & ... & 251 & 251 & 251 & 251 \\
251 & 251 & 251 & 251 & ... & 251 & 251 & 251 & 251 \\
251 & 251 & 251 & 251 & ... & 251 & 251 & 251 & 251 \\
217 & 217 & 217 & 217 & ... & 217 & 217 & 217 & 217
\end{bmatrix}
$$

The numeric entries merely represent some color value.

Arnold's Cat Map

The particular instance of chaos we will explore in this discussion is the chaotic mapping called Arnold's cat map in recognition of Russian mathematician Vladimir I. Arnold, who discovered it using an image of a cat. It is a simple and elegant demonstration and illustration of some of the principles of chaos – namely, underlying order to an apparently random evolution of a system. An image (not necessarily a cat) is hit with a transformation that apparently randomizes the original organization of its pixels. However, if iterated enough times, as though by magic, the original image reappears.

If we let $X = \begin{bmatrix} x \\ y \end{bmatrix}$ be a n x n matrix of some image, Arnold's cat map is the transformation

$$
\boxed{\Gamma \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x+y \\ x+2y \end{bmatrix} \bmod n}
$$

where mod is the modulo of the $\begin{bmatrix} x+y \\ x+2y \end{bmatrix}$ and n . For example, $3.142 \bmod 1 = .142$ or $150 \bmod 100 = 50$ or $\begin{bmatrix} 123 \\ 154 \end{bmatrix} \bmod 100 = \begin{bmatrix} 23 \\ 54 \end{bmatrix}$. Since the signs of both arguments are the same sign in this exercise, the modulo will simply be the remainder of the long division of $\begin{bmatrix} x+y \\ x+2y \end{bmatrix}$ and n.

To better understand the mechanism of the transformation Γ, let us decompose it into its elemental pieces.

1. Shear in the x-diection by a factor of 1.

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x+y \\ y \end{bmatrix}$$

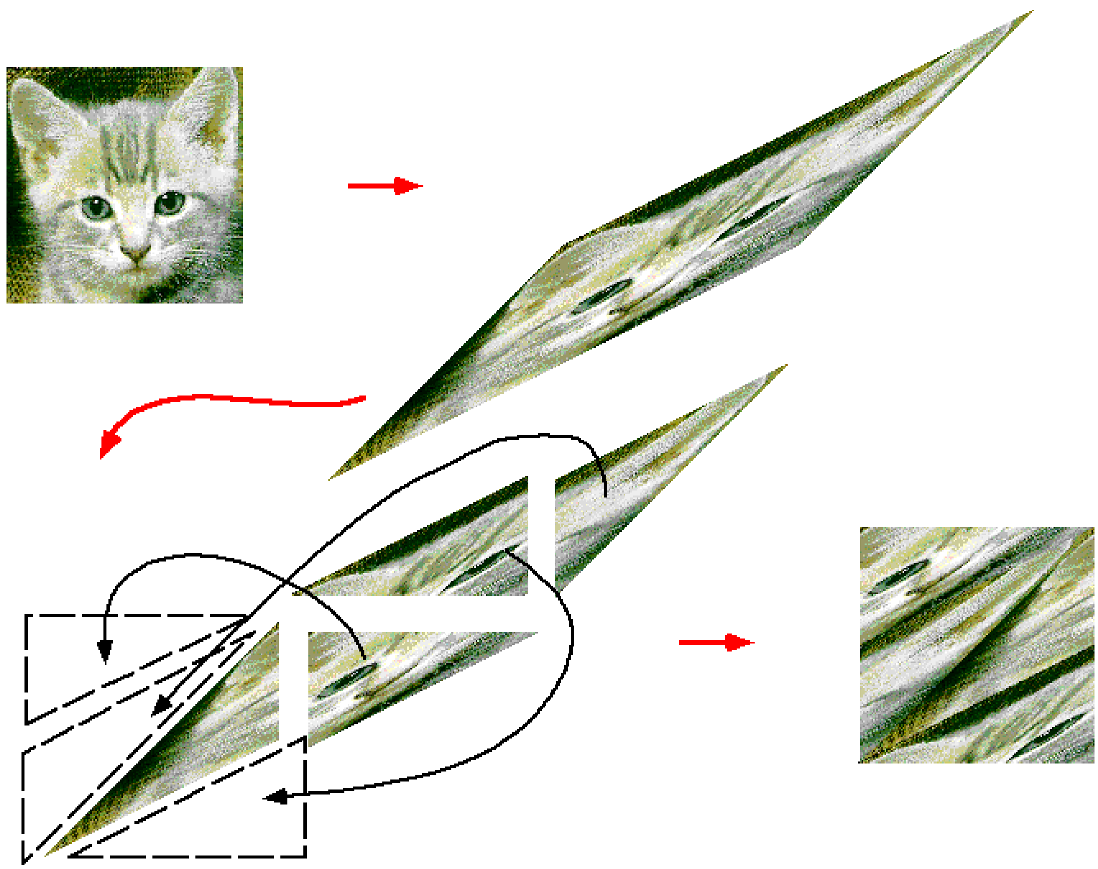
2. Shear in the y-direction by a factor of 1.

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ x+y \end{bmatrix}$$

3. Evaluate the modulo.

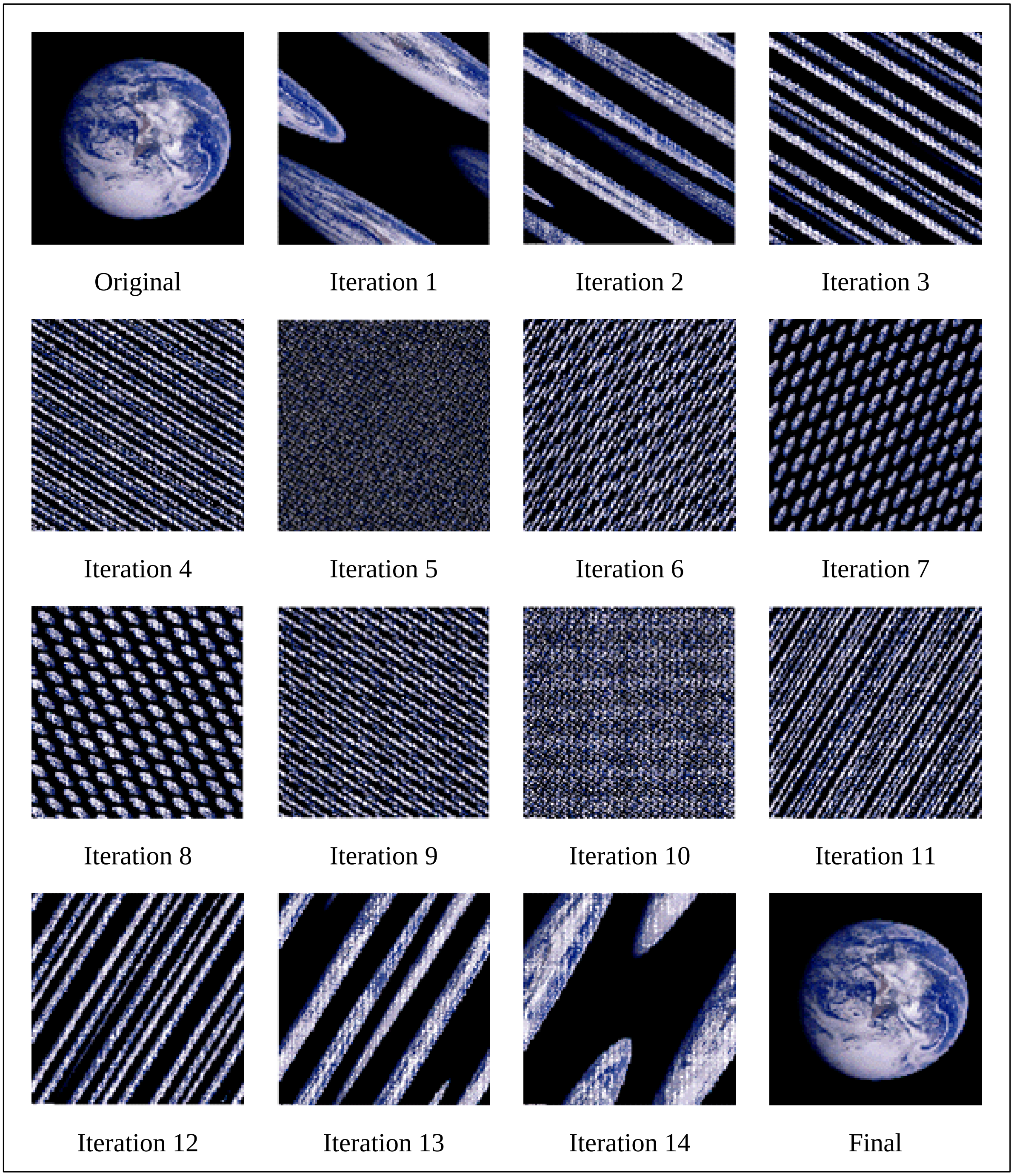
$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ y \end{bmatrix} \bmod n$$

Included below is a visual aide illustrating these steps. The first step shows the shearing in the x- and y-directions, followed by evaluation of the modulo and reassembly of the image.



The Experiment

Having developed the mathematical tools to address the heart of this discussion, an experiment may be performed to determine if, indeed, there exists any order – underlying or otherwise – to the chaotic mapping of Arnold's cat map. For this purpose, a program was developed for MATLAB by the author (see accompanying program "catbox.m"), and the following $124 \; x \; 124$ image of the Earth was iterated with the transformation Γ.



The pixels rapidly degenerate into a television-static of chaos by iteration number five, with some unintelligible order prominent in a number of iterations prior to the original image reappearing on the fifteenth iteration. Therefore, the image is said to have a period of fifteen. That periodicity should be observed in a system such as this is quite extraordinary. It's not unlike a coherent image spontaneously leaping out of the salt-and-pepper of television static, a cogent message spoken from the howl of radio static – loaded dice, indeed!

Experimentally, no elegant model could be developed for the relationship between the period of an image and *n*, its number of rows or columns. In general, it may be claimed that as the value of *n* increases, the period tends to increases. However, this is not always true. For example, a $101\ x\ 101$ image has a period of twenty-five; whereas, a $124\ x\ 124$ image, as we just learned, has a period of fifteen. Other luminaries have found a relationship where this experimenter failed – but it certainly cannot be claimed to be elegant nor robust . Let the period be $\prod$.

1. $\prod(n) = 3n$ if and only if $n = 2 \bullet 5^k$ for $k = 1, 2, \ldots$

2. $\prod(n) = 2n$ if and only if $n = 5^k$ or $n = 6 \bullet 5^k$ for $k = 1, 2, \ldots$

3. $\prod(n) \leq \frac{12n}{7}$ for all other choices of n

Let us address that ageless question of Mankind: Why? Why does order emerge out of this apparently chaotic mapping? The best approach to answer this question is, perhaps, to examine the behavior of a single pixel, to determine what effect Arnold's cat map has upon it. Consider the ordinary pixel $\begin{bmatrix} 52 \\ 13 \end{bmatrix}$ of the $124\ x\ 124$ image considered previously. It takes the following path:

$$\Gamma\begin{bmatrix} 32 \\ 13 \end{bmatrix} = \begin{bmatrix} 32 & + & 13 \\ 13 & & 2 \bullet 32 \end{bmatrix} \bmod 124 = \begin{bmatrix} 45 \\ 58 \end{bmatrix} \to \begin{bmatrix} 103 \\ 37 \end{bmatrix} \to \begin{bmatrix} 16 \\ 53 \end{bmatrix} \to \begin{bmatrix} 69 \\ 122 \end{bmatrix}$$

$$\to \begin{bmatrix} 67 \\ 65 \end{bmatrix} \to \begin{bmatrix} 8 \\ 73 \end{bmatrix} \to \begin{bmatrix} 81 \\ 30 \end{bmatrix} \to \begin{bmatrix} 111 \\ 17 \end{bmatrix} \to \begin{bmatrix} 4 \\ 21 \end{bmatrix} \to \begin{bmatrix} 25 \\ 46 \end{bmatrix} \to \begin{bmatrix} 71 \\ 117 \end{bmatrix}$$

$$\to \begin{bmatrix} 64 \\ 57 \end{bmatrix} \to \begin{bmatrix} 121 \\ 54 \end{bmatrix} \to \begin{bmatrix} 51 \\ 105 \end{bmatrix} \to \begin{bmatrix} 32 \\ 13 \end{bmatrix}$$

After fifteen iterations, the pixel – as would any other pixel in the image – has returned to its initial position, and it would continue eternally along this circle if iterated accordingly. This agrees with the earlier observation that the $124\ x\ 124$ image has a period of fifteen.

Conclusions

In closing, it is important to consider the question that invariably and naturally occurs: What is the utility, what are the practical applications of Arnold's cat map? Taken by itself, the utility is limited. However, if taken as a small piece in the whole gestalt of chaos; as an illustration of the power of this new science; as a glimpse into the machinery of the Universe; or as evidence that God does play dice with the Universe, but they are loaded dice – the utility is immense. Chaos gives us the tools to approach mathematical and physical phenomena – namely, complex, chaotic systems – with confidence – or at least competence – where before we were at a loss. Examples of chaos in Nature abound: the beating of a heart, the eccentricity of the planet Pluto's orbit, the Great Red Spot of Jupiter, the turbulence of fluids, the capricious weather, the swinging of a pendulum, the formation of a snowflake... Chaos brings us another step closer in our long journey to understanding the Universe we live in.

Acknowledgments

In writing this, the following texts were invaluable: **Linear Algebra and Applications** by Howard Anton; **Chaos and Fractals** by Heinz-Otto Peitgen, Hartmut Jurgens, and Dietmar Saupe; and **Chaos**: **Making a New Science** by James Gleick. For anyone interested in an introduction the chaos, the latter two texts are highly recommended. I would like to acknowledge Leon Poon of the University of Maryland for generously allowing me to make use of the graphic illustrating the different steps involved in one iteration of Arnold's cat map; Dr. David Mills of College of the Redwoods for guiding me to this project; and, of course, David Arnold of College of the Redwoods, without whose patient assistance and guidance while developing the companion program to this paper (”catbox.m”), this project would not have been possible.

References

Anton, Howard, **Linear Algebra and Applications 7th Edition**. New York, New York: John Wiley and Sons, 1994.

Gleick, James, **Chaos**: **Making a New Science**. New York, New York: Penguin Books, 1988.

Peitgen, Heinz-Otto, Jurgens, Hartmut, Saupe, Dietmar, **Chaos and Fractals**. New York, New York: Springer-Verlag, 1992.