Fourier: An Introduction From Series To Transforms

Randy Cessna and Lori Williford
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Abstract

We introduce Fourier analysis beginning with a look into how the Fourier series is derived, and how to get the Fourier coefficients. We then will discuss the Fourier transforms and some useful applications.

1 Introduction

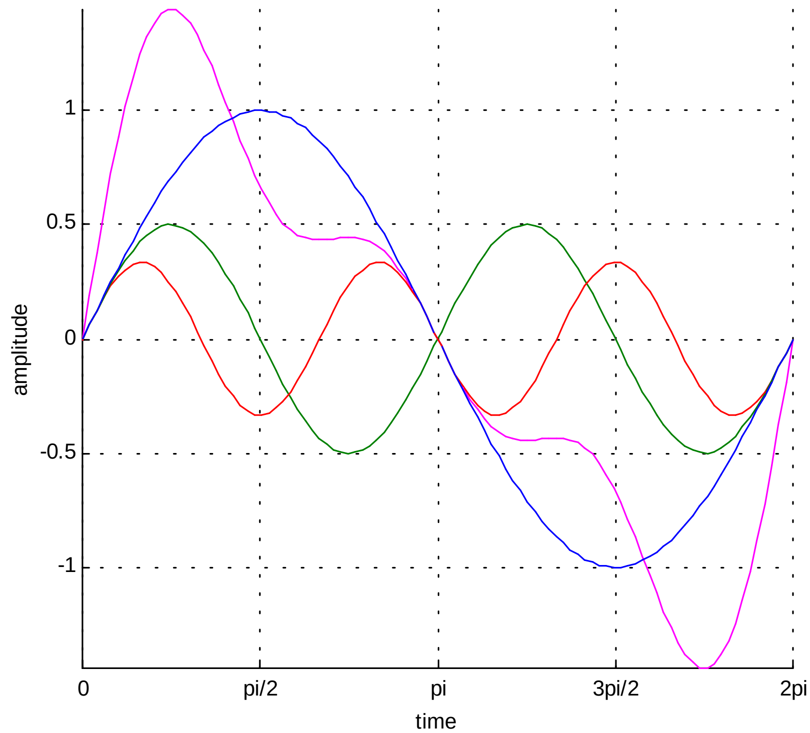
John Baptiste Fourier was a French mathematical physicist who was interested in the flow of heat through objects. Heat travels in waves, which are periodic functions, and Fourier made a profound observation. He observed that any complicated wave can be made up of the sums of simpler waves. This intelligent insight has revcolutioned modern scientific data analysis and now allows easier and quicker understanding of any periodic function.

1.1 Understanding the Fourier Series

To understand the Fourier series you will need to recall some trigonometry. The general form of a sine or cosine equation follows:

$$a\cos\omega t$$
$$b\sin\omega t$$

where a,b are the ampliudes, ω is the angular velocity, which is defined to be $2\pi * f$, f is $\frac{1}{T}$, and T is the period with these equatons, we can understand every part of the wave. If we add up a combination of sine waves with different amplitudes and frequencies, then we would have a complicated wave



where the blue wave is $sint$, the green wave is $2sin2t$, the red wave is $3sin3t$, and the magenta wave is $sint + 2sin2t + 3sin3t$.We could also add cosine waves together or add sine and cosine waves together , which is the basis for the fourier series

$$\begin{aligned} f(t) &= \frac{a_0}{2} + a_1 \cos\left(\frac{2\pi t}{T}\right) + a_2 \cos\left(\frac{2(2)\pi t}{T}\right) + a_3 \cos\left(\frac{2(3)\pi t}{T}\right) + ... \\ & +a_k \cos\left(\frac{2k\pi t}{T}\right) + ... + b_1 \sin\left(\frac{2\pi t}{T}\right) + b_2 \sin\left(\frac{2(2)\pi t}{T}\right) + \\ & b_3 \sin\left(\frac{2(3)\pi t}{T}\right) + ... + b_k \sin\left(\frac{2k\pi t}{T}\right) + ... \end{aligned}$$

which can be written in shorter summation form

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2k\pi t}{T}\right) + b_k \sin\left(\frac{2k\pi t}{T}\right)$$

where a_0, a_k, b_k are known as the fourier coefficients.

1.1.1 Orthogonality

To understand how to derive the fourier coefficients we will need to recall some linear algebra. Recall that a basis is defined to be the set of linearly independent vectors which span a subspace of R. A vector space is defined to be a nonempty set V of vectors, on which are defined two operations,addition and scalar multiplication, subject to the ten axioms listed below.

1. The sum of u and v, denoted by $u + v$, is in V
2. $u + v = v + u$
3. $(u + v) + w = u + (v + w)$
4. There is a *zero* vector 0 in V such that $u + 0 = u$
5. For each u in V, there is a vector $-u$ in V such that $u + (-u) = 0$
6. The scalar multiple of u by c, denoted by cu, is in V
7. $c(u + v) = cu + cv$
8. $(c + d)u = cu + du$
9. $c(du) = (cd)u$
10. $1u = u$

An inner product on V would be:

$$\langle f, g\rangle = \int_0^T f(t)g(t)dt$$

which has the following properties

1. $u \cdot v = v \cdot u$
2. $(u + v) \cdot w = u \cdot w + v \cdot w$
3. $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$
4. $u \cdot u \geq 0$, and $u \cdot u = 0$ if and onlt if $u = 0$

So let V be a vector space such that:

$$V = \{f : R \to R : f \text{ has period } T\}$$

A basis for V would be:

$$\beta = \left\{ \begin{array}{c} 1, \cos(\frac{2\pi t}{T}), \cos(\frac{2(2)\pi t}{T}), \cos(\frac{2(3)\pi t}{T}), ..., \cos(\frac{2k\pi t}{T}), ... \\ , \sin(\frac{2\pi t}{T}), \sin(\frac{2(2)\pi t}{T}), \sin(\frac{2(3)t}{T}), ..., \sin(\frac{2k\pi t}{T}), ... \end{array} \right\}$$

which is an infinite dimensional basis, unlike the small and finite basis that we encountered in our linear algebra class. It is also important to note that this is also an orthogaonal basis for almost any entry contained within.

Finding the Fourier coefficients To develop the formulas for the fourier coefficients we need to first look at some inner product spaces.

$$\left\langle \cos\left(\frac{2m\pi t}{T}\right), \cos\left(\frac{2n\pi t}{T}\right)\right\rangle = 0, m \neq n$$

Using the trig identity $cosAcosB = \frac{1}{2}\left\{\cos(A+B) + \cos(A-B)\right\}$ you get

$$\int_0^T \cos\left(\frac{2m\pi t}{T}\right) \cos\left(\frac{2n\pi t}{T}\right) dt = 0$$

Therefore,

$$\left\langle \sin\left(\frac{2m\pi t}{T}\right), \sin\left(\frac{2n\pi t}{T}\right)\right\rangle = 0, m \neq n$$

$$\left\langle \cos\left(\frac{2m\pi t}{T}\right), \sin\left(\frac{2m\pi t}{T}\right)\right\rangle = 0$$

$$\left\langle \cos\left(\frac{2m\pi t}{T}\right), \cos\left(\frac{2m\pi t}{T}\right)\right\rangle = \frac{T}{2}$$

$$\left\langle \sin\left(\frac{2m\pi t}{T}\right), \sin\left(\frac{2m\pi t}{T}\right)\right\rangle = \frac{T}{2}$$

where the only non-orthogonal products are integer multiples of the same frequencies. Following the same technique you can see that

$$\begin{aligned}
\left\langle f, \cos\left(\frac{2m\pi t}{T}\right)\right\rangle &= a_k \left\langle \cos\left(\frac{2m\pi t}{T}\right), \cos\left(\frac{2m\pi t}{T}\right)\right\rangle \\
\left\langle f, \cos\left(\frac{2m\pi t}{T}\right)\right\rangle &= a_k * \frac{T}{2} \\
a_k &= \frac{2}{T}\left\langle f, \cos\left(\frac{2m\pi t}{T}\right)\right\rangle \\
a_k &= \frac{2}{T}\int_0^T f(t) \cos\left(\frac{2m\pi t}{T}\right) dt
\end{aligned}$$

Finding b_k and a_0 involve the same process and you arrive at

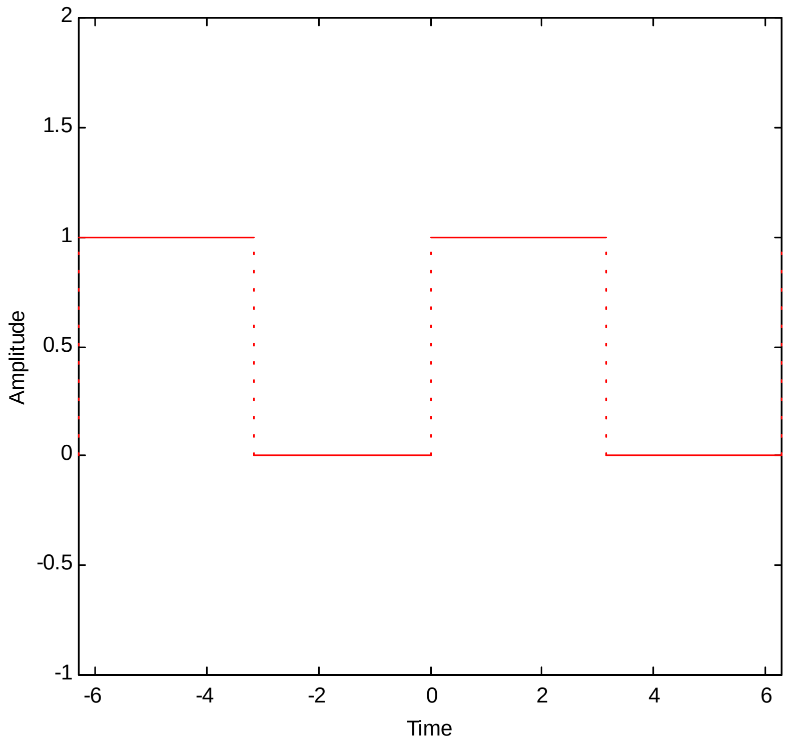
$$b_k = \frac{2}{T}\int_0^T f(t) \sin\left(\frac{2m\pi t}{T}\right) dt$$

$$a_0 = \frac{1}{T} \int_0^T f(t)dt$$

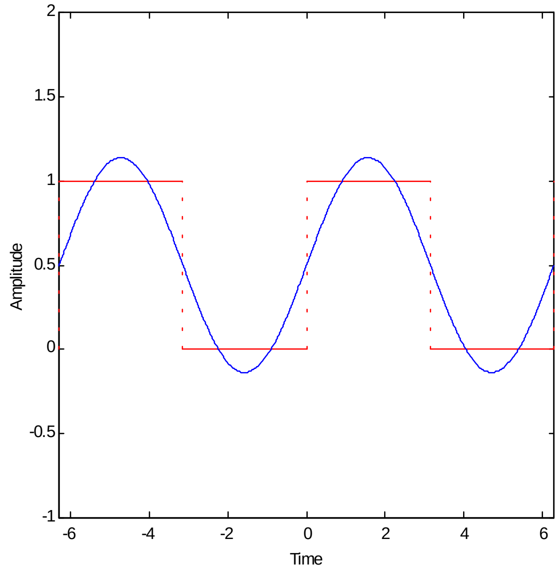
Now we have formulas for the fourier series, and all the fourier coefficients and it is time for a demonstration.

Example 1 *Let's say we have an electrical current that is being swityched on and off. It can be described by the function*

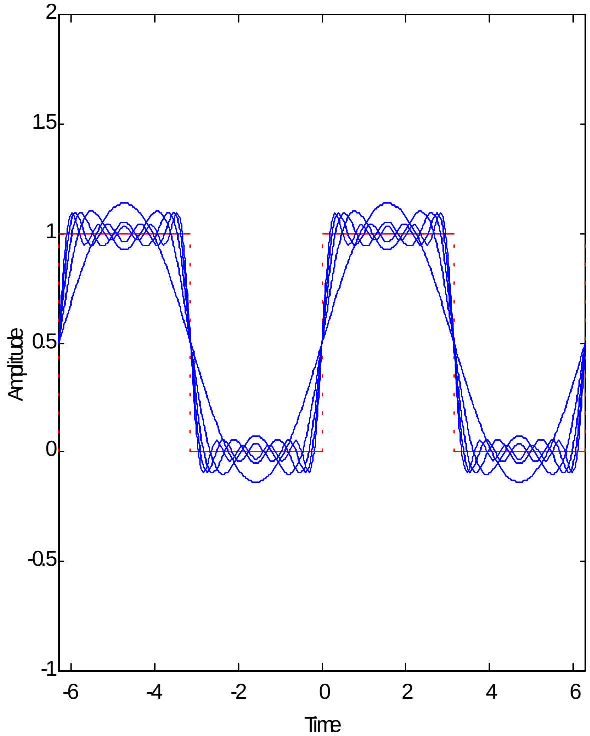
$$f(t) = \begin{cases} 1 & if \quad -2\pi \leq t < -\pi \\ 0 & if \quad -\pi \leq t < 0 \\ 1 & if \quad 0 \leq t < \pi \\ 0 & if \quad \pi \leq t < 2\pi \end{cases}$$



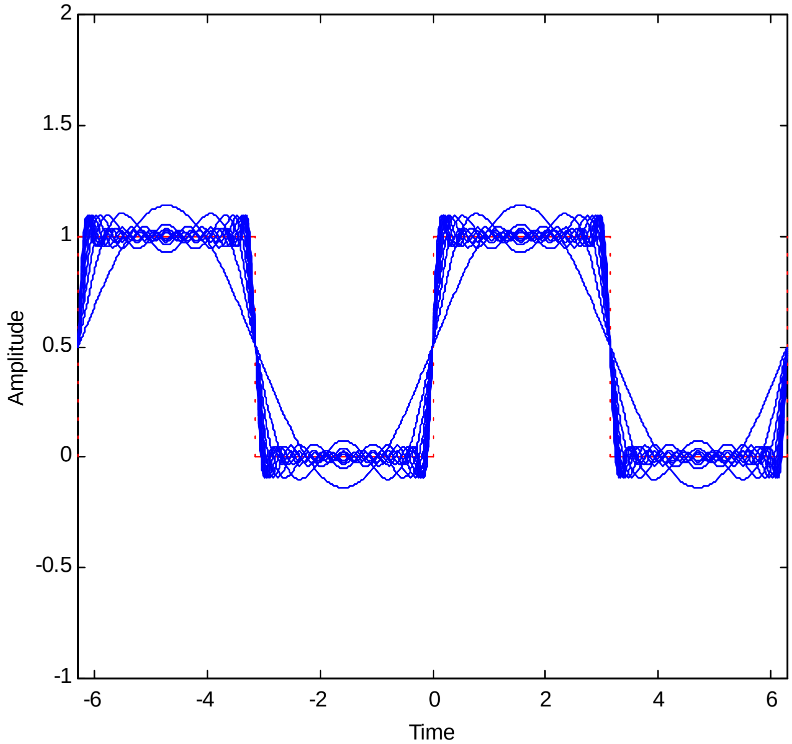
lets use only one tertm in our approximation and see how it works



Since this is an odd function, we will not be using any of the even harmonics(terms in the fourier series)since they will all equal zero. Now lets look at an approximationn that goes to the ninth harmonic.



And one more approximation with 19 terms.



As you can see, each succesive approximation comes closer and closer to approximating the square wave function. If allowed to have an infinite number of harmonics, the fourier expansion would be an excellent global approximation of the complicated function except for the very edges of the square wave. This effect is known as the Gibb's phenomena, and would be slightly off at the ends of the piecewise defined square wave function.

1.2 Fourier Transform

The Fourier series and the Fourier integral are separate and distinct yet a subtle relationship exists between the two. A Fourier series can exist if it meets several conditions, The first of these is that the waveform must be periodic. Periodic technically means that the waveform starts at minus infinity and repeats itself to positive infinity. But for our case periodicity will be defined for the finite time interval. Also three other conditions are required these are commonly referred to as Dirichlet's conditions.

- The number of discontinuity in the modeled waveform must be finite over the observed period.

- The wave form may also only have a finite number of max and minima during that period of observation.
- The function call it $f(t)$ has to be completely integratable for all t.

$$\int_0^T abs[x(t)]dt\infty$$

If all these requirements are satisfied then the waveform x(t) can be approximated with the Fourier series.

$$x(t) = a_0 + \sum_{\nabla=1}^{\infty} (a_n \cos n\varpi_0 t + b_n \sin n\varpi_0 t)$$

Of course it is impossible to make an infinite degree series. So truncating a series to get a good approximation is its most practical usage. Error is also included with modeling discontinuous functions. This error is commonly referred to as the Gibb's effect and will never disappear even with an infinity Fourier series.

Fourier analysis is the prelude for spectral analysis. In chemistry, chemical elements are often distinguished from one another by their line spectrum, the sun and stars are each characterizable by the light each gives off. Likewise different waveforms have different spectra,components of amplitude, frequency, and phase, that are unlike any other waveform's spectra.

Any natural occillating periodic function in nature is distinguishable by its own particular frequency spectrum and can be analyisised with Fourier series analysis. But not all waveforms in nature are periodic some are of infinite period. A waveform that does not repeat itself over is considered to have infinite period. Such waveforms require the Fourier integral for analysis.

The Fourier integral can be derived from the Fourier series in four important steps.

1. The series is written in its exponential form using the exponential identities for cosine and sine.

$$\cos 2\pi n f_0 t = \left(\frac{e^{j2\pi n f_0 t} + e^{-j2\pi n f_0 t}}{2} \right)$$

$$\sin 2\pi n f_0 t = \left(\frac{e^{j2\pi n f_0 t} - e^{-j2\pi n f_0 t}}{2j} \right)$$

   where e is the base of the natural logarithm and j is the imaginary unit of the complex number system.

2. The exponential form of the Fourier series is more compact.

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}$$

where c_n is evaluated for $n = -\infty, ..., -2, -1, 0, 1, 2, ..., \infty$ by

$$c_n = \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} x(t)\, e^{-j2\pi n f_0 t} dt$$

For each n, c_n is evaluated to give the magnitude and phase of the harmonic component of $x(t)$ having frequency nf_0.

3. Since $\Delta f = \frac{1}{T}$, the series can be put into a form that allows inspection of the limit as T goes to infinity.

$$x(t) = \frac{\lim}{\Delta f \to 0} \sum_{n=-\infty}^{\infty} X(nf_0)\, e^{j2\pi n f_0 t \Delta f}$$

4. As Δf goes to zero the properties of the summation approach those of an integral.

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t} df$$

This last integral is the Fourier transform. The Fourier transform has at least nine important properties associated with its importance.

1. The Fourier transform is invertible.

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t} dt$$

2. Evenfunctions transform to real parts only.
3. Odd functions transform to imaginary parts only.
4. Arbitrary functions are the sum of even and odd parts.
5. The linearity property is satisfied.

$$x(t) + y(t) \text{ transform to } X(f) + Y(f)$$

6. Time scaling affects frequency and amplitude scaling.
7. Frequency scaling affects time and amplitude scaling.

8. Time shifting affects phase only.

9. Frequency shifting causes time domain modulation.

If a waveform can be mathematically described then the Fourier transform can be used, however the accompanying mathematics can be really steep. Many common waveforms can be found in pre-existing tables and charts. However finding a match their for a naturally occurring wave is rare. So this is where waveform digitizing and digital signal processing comes in. The discrete Fourier transform (DFT) and the fast Fourier transform (FFT). The FFT is simply an efficient algorithm for computing the DFT.

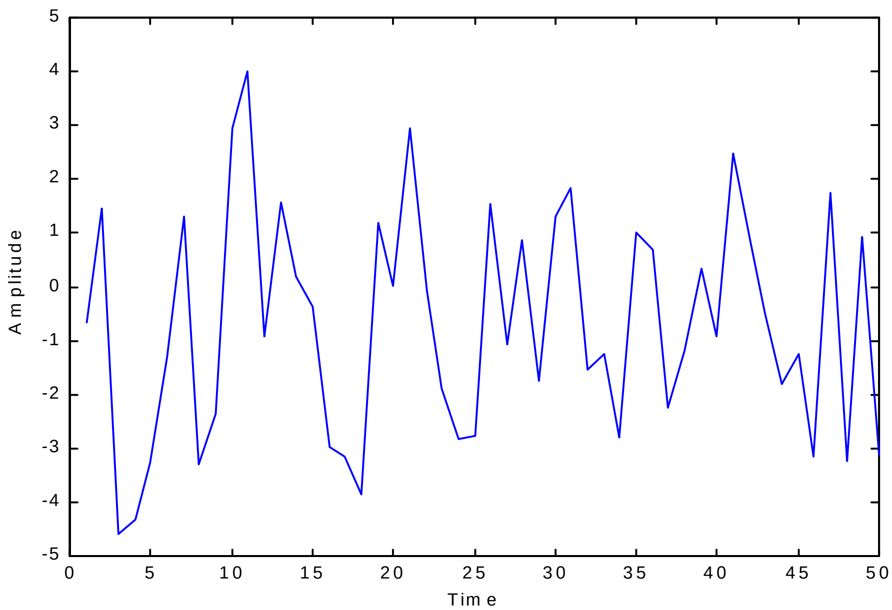
The DFT has a formula.

$$X_d\left(k\Delta f\right) = \Delta t \sum_{n=0}^{N-1} x\left(n\Delta t\right) e^{-j2\pi k\Delta f n\Delta t}$$

This expression allows you to transform a time series of samples, such as a table of data points, to a series of frequency-domain samples. Similarly like the Fourier integral the DFT also has an inverse.

$$\begin{aligned}
N &= \text{ number of samples being considered} \\
\Delta t &= \text{ the sampling interval, from this } N\Delta t \text{ gives the time record length} \\
\Delta f &= \text{ the sample interval in the frequency domain and } = \frac{1}{N\Delta t} \\
n &= \text{the time sample index where } \; n = 0, 1, 2, ..., N-1 \\
k &= \text{the index set for discrete frequency components } k = 0, 1, 2, ..., N-1 \\
x\left(n\Delta t\right) &= \text{the discrete set of time samples defines the waveform to be transformed} \\
X(k\Delta f) &= \text{the set of fourier coefficients obtained by the DFT of } x\left(n\Delta t\right) \\
e &= \text{the base of the natural logarithm} \\
j &= \text{the symbol of complex notation, } \; j = \sqrt{-1}
\end{aligned}$$

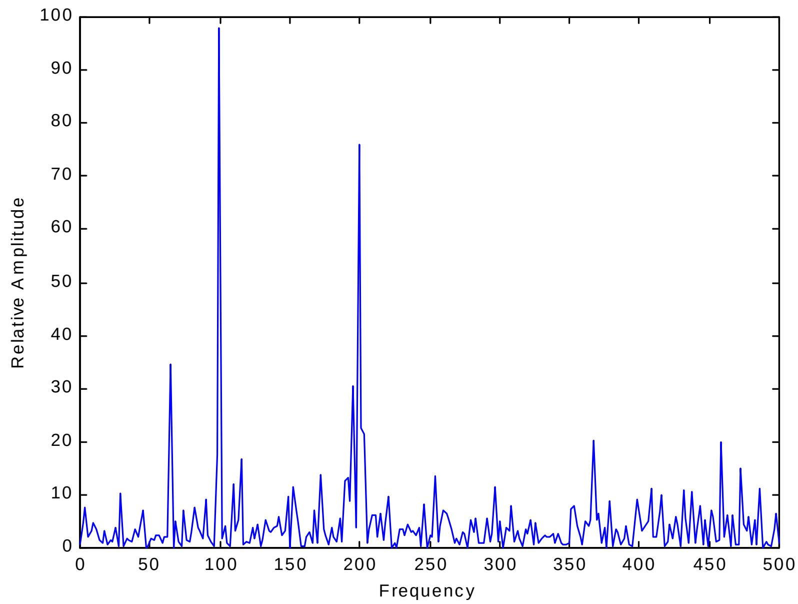
1.2.1 An example



$$f(t) = \cos(200\pi t) + \cos(400\pi t) \text{ and background noise}$$

This graph consists of a signal that is buried in noise. It is difficult to indicate any underlying frequency just by this graph. The DFT can transform the time domain signal to the frequency domain.

Matlab has a powerful FFT algorithm built in.



This last graph was produced by using matlabs FFT command and sampling 512 points of the original signal.

In conclusion, the FFT has many applications in many fields of discipline, such as chemical engineering, electrical engineering, and many other areas of study. Practically anything that occilates, shakes, rattles or rolls with respect to time can be analized with Fourier techniques.

1.3 References

- *Elementary Differential Equations And Boundary Value Problems* by William E. Boyce and Richard C. DiPrima
- *Electric Circuits* by James W. Nilsson and Susan A. Riedel
- *The FFT Fundamentals and Concepts* by Robert W. Ramirez
- *Who is Fourier* by international college of language