Forest Management—Sustainable Harvesting

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1 Forest Management

Most of our economy in the North Coast is directly or indirectly related to the timber industry. Many of us take an interest in this business as it impacts our livelihoods. We have seen different types of forest harvesting and there has been concern over some of those harvesting policies. This model by Howard Anton and Chris Rorres was especially appealing to us as it presents a very basic model for sustainable harvesting of forests using some elementary concepts of linear algebra.

1.1 Discussion

The ultimate objective in this model is to arrive at the maximum amount of money a forest is able to yield while retaining its initial configuration. This is called the optimal sustainable yield. To arrive at the optimal sustainable yield there is a condition that must be satisfied. This condition mathematically insures that the forest will replace every tree harvested with a new tree. This condition also accounts for the growth factors of each height class. This condition takes the initial number of trees, the growth factor, the number of trees harvested, and the number of trees replanted forest, applies linear algebra and returns the forest to its exact configuration. This means that the total number of trees will always be the same, as well as, the total number of trees in each height class will always remain the same. This condition is called the sustainable harvesting condition and is figured using the tools of linear algebra.

After analyzing this model we realized that the likelihood of this model being applied to an existing forest was remote. The restrictions for this model that make the sustainable harvesting condition feasible seem to require a forest with a particular initial configuration, one that may need to be preplanned, a tree farm for example. We consulted with Ross Tomlin of The College of the Redwoods Forestry Department. After a couple of visits we all decided that this model, as is, could possibly be used for tree farms, most likely Christmas tree farms. The limitations and over simplification of the model restricts it from being used in a naturally occurring forest. Because this is an over simplified model we realize that many more complex factors would eventually be worked in that would lift the restrictions and account for a variance from forest to forest. With this in mind we realize that understanding this model gives us a basis for comprehending a much more complex model of forestry management.

1.2 Assumptions and Restrictions of This Model

1. The total number of trees in the forest will always be the same. The total number of trees in each height class will also remain the same. The forest will undergo a cyclic pattern of growth, harvest, and replanting. Each time this process is completed the forest is returned to its original configuration. The forest will be divided into different height classes. Each height class will correspond to a respective entry in each vector.

2. This model does not take into account trees that die.

3. A tree can move at most up one height class.

4. The height of each tree will determine the economic value when the trees are cut down and sold.

1.3 Variables For the Sustainable Harvesting Condition

1.3.1 The Nonharvest Vector

Let the vector $\mathbf{x}$ represent the number of trees in each class. Each entry in $\mathbf{x}$ represents the number of trees in each height class. We will refer to $\mathbf{x}$ as the nonharvest vector.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

For a sustainable harvesting policy, the forest must be returned to the fixed configuration given by the nonharvest vector x.

1.3.2 The Harvest Vector

Let the vector y be the total number of trees cut from the forest. Each entry in y represents the number of trees harvested from each height class. We will refer to $\mathbf{y}$ as the harvest vector.

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

1.4 Matrices and Their Variables

In this model trees can only do two things:

- - grow into the next height class
  - remain in the same height class

The percentages of growth of each class will be placed in a matrix, appropriately called The Growth Matrix.

1. Let g_i be the fraction of trees in the ith class that grow into the next class.
   - This percentage will be placed on the -1 diagonal of the matrix
2. Therefore, $1-g_i$ represents the fraction of trees in the ith class that remain in the ith class.
   - This percentage will be placed on the main diagonal.

The Growth Matrix is comprised of only those values, g, and $g-1$. Multiplying the numerical values of the growth matrix with the vector x will give us the number of trees in each height class after a growth period. This value $G\mathbf{x}$ will later determine the number of trees to be harvested from each height class.

1.4.1 Growth Matrix

$$G = \begin{bmatrix} 1-g_1 & 0 & 0 & & \cdots & 0 \\ g_1 & 1-g_2 & 0 & & \cdots & 0 \\ 0 & g_2 & 1-g_3 & & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & 0 \\ 0 & 0 & 0 & \cdots & 1-g_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & g_{n-1} & 0 \end{bmatrix}$$

$$G\mathbf{x} = \begin{bmatrix} (1-g_1)x_1 \\ g_1x_1 + (1-g_2)x_2 \\ g_2x_2 + (1-g_3)x_3 \\ \vdots \\ g_{n-2}x_{n-2} + (1-g_{n-1})x_{n-1} \\ g_{n-1}x_{n-1} + x_n \end{bmatrix}$$

1.4.2 Replacement Matrix

Because the forest will be replanted with the same number of trees that were harvested, the harvest vector also tells us how many new seedlings will need to be planted. To keep the harvesting and replacement values distinct, a replacement matrix is formed. The vector **y** represents the number of trees to be harvested, and R**y** represents the number of seedlings to be planted after the harvest.

$$R = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} R\mathbf{y} &= \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \\ &= \begin{bmatrix} y_1 + y_2 + \cdots + y_n \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{aligned}$$

1.5 Sustainable Harvesting Condition

In words,

$$\begin{bmatrix} \text{configuration} \\ \text{at end of} \\ \text{growth period} \end{bmatrix} - \begin{bmatrix} \text{trees} \\ \text{harvested} \end{bmatrix} + \begin{bmatrix} \text{new seedling} \\ \text{replacements} \end{bmatrix} = \begin{bmatrix} \text{configuration} \\ \text{at beginning of} \\ \text{growth period} \end{bmatrix}$$

Or mathematically,

$$G\mathbf{x} \ - \ \mathbf{y} \ + \ R\mathbf{y} \ = \ \mathbf{x}$$

Regrouping the variables:

$$G\mathbf{x} \ - \ \mathbf{x} \ = \ \mathbf{y} \ - \ R\mathbf{y}$$

Factoring out the vectors and using the Identity Matrix:

$$(G \;-\; I)\mathbf{x} \;=\; (I \quad - \;R)\mathbf{y}$$

$$\left(\begin{bmatrix} 1-g_1 & 0 & 0 & & \cdots & 0 \\ g_1 & 1-g_2 & 0 & & \cdots & 0 \\ 0 & g_2 & 1-g_3 & & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & 0 \\ 0 & 0 & 0 & \cdots & 1-g_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & g_{n-1} & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

$$= \left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}\right) \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix}$$

$$\begin{bmatrix} -g_1 & 0 & 0 & \cdots & 0 & 0 \\ g_1 & -g_2 & 0 & \cdots & 0 & 0 \\ 0 & g_2 & -g_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -g_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & g_{n-1} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_4 \\ x_5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & -1 & \cdots & -1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_4 \\ y_5 \end{bmatrix}$$

Multiplying each side of the equation out and setting the terms equal to each other:

$$\begin{aligned} y_1 &= 0 \\ y_2 + y_3 + \cdots + y_5 &= g_1 x_1 \\ y_2 &= g_1 x_1 - g_2 x_2 \\ y_3 &= g_2 x_2 - g_3 x_3 \\ &\vdots \\ y_{n-1} &= g_{n-2} x_{n-2} - g_{n-1} x_{n-1} \\ y_n &= g_{n-1} x_{n-1} \end{aligned}$$

The first value for the first height class of the harvest vector (y_1) will always be equal to zero, for, of course, none of the seedlings will be harvested. This outcome says that :

- The total number of trees harvested equals the number of trees that grew from the first height class to the second height class.
- The number of trees to be harvested in each individual height class will equal the number of trees that grew into a particular height class minus the number of trees that grew out of that same height class. This is how the respective height classes retain their initial configuration.

In order for each value of **y** to be positive, for there can be no negative number of trees harvested, the percentage of growth of each lower height class minus the next larger height class must be positive. In other words, the number of each trees after a growth period in a respective height class must be higher than the number of trees that grew into the next height class. Therefore:

$$g_1x_1 \geq g_2x_2 \geq \cdots \geq g_{n-1}x_{n-1} \geq 0$$

This condition, fundamental to the sustainable yield condition, is also the one that restricts the application from being applied to an existing forest.

1.6 Variables for the Optimal Sustainable Yield

Let s be the total number of trees in the forest. This value is found by adding all the values in the non harvest vector **x**:

$$s \quad = x_1 + x_2 + \cdots + x_n$$

Let the vector **p** represent the price in dollars paid for a tree in each respective height class.

$$\mathbf{p} \ = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

To find a total yield in terms of dollars, multiply the number of trees harvested from each class, **y**, by the price paid for each tree in that class, **p**, and total the sums:

$$Yld = p_2y_2 + p_3y_3 + \cdots + p_ny_n$$

Because each y_i term can be defined in terms of $g_ix_i - g_{i+1}x_{i+1}$, these values can be substituted in for each value in **y:**

$$Yld = p_2(g_1x_1 - g_2x_{2)} + p_3(g_2x_2 - g_3x_3) + ... + p_n(g_{n-1}x_{n-1} - g_nx_n)$$

After multiplying this out and regrouping the p_i values:

$$Yld = p_2 g_1 x_1 + (p_3 - p_2)\, g_2 x_2 + \cdots + (p_n - p_{n-1})\, g_{n-1} x_{n-1}$$

1.7 Theorem 1: The optimal sustainable yield is achieved by harvesting all the trees from one particular height class and none of the other height classes.

This theorem is debatable and is not one currently held by Ross Tomlin in the forestry department, but it is a condition for this particular model.

$$Yld_k = \text{the only class to be harvested}$$

None of the other height classes will be harvested. This is said mathematically by:

$$y_2 = y_3 = \cdots = y_{k-1} = y_{k+1} = \cdots = y_n = 0$$

Because the total number of trees harvested is now y_k, we can substitute y_k for $y_2 + y_3 + \cdots + y_n$ in the sustainable yield condition previously arrived at. Because none of the other height classes will have trees harvested from them, we have to put a zero in place of the y_i values. The sustainable yield condition now looks like this:

$$\begin{aligned}
y_k &= g_1 x_1 \\
0 &= g_1 x_1 - g_2 x_2 \\
0 &= g_2 x_2 - g_3 x_3 \\
&\vdots \\
0 &= g_{n-2} x_{n-2} - g_{n-1} x_{n-1} \\
y_k &= g_{n-1} x_{n-1}
\end{aligned}$$

By adding each negative $g_i x_i$ term to the zero on the other side of the equation it follows that:

$$y_k = g_1 x_1 = g_2 x_2 = \cdots = g_{n-1} x_{n-1}$$

After setting each $g_i x_i$ value equal to the next, and then solving for each x_i it follows that:

$$\begin{aligned}
x_2 &= \frac{g_1 x_1}{g_2} \\
x_3 &= \frac{g_1 x_1}{g_3} \\
&\vdots \\
x_{k-1} &= \frac{g_1 x_1}{g_{k-1}}
\end{aligned}$$

By taking the value for the total number of trees in the forest:

$$s = x_1 + x_2 + \cdots + x_n$$

and substituting in the new values for the each value of $\mathbf{x}$ we get the following:

$$s = x_1 + \frac{g_1 x_1}{g_2} + \frac{g_1 x_1}{g_3} + \cdots + \frac{g_1 x_1}{g_{k-1}}$$

Factor:

$$s = x_1 \left(1 + \frac{g_1}{g_2} + \frac{g_1}{g_3} + \cdots + \frac{g_1}{g_{k-1}} \right)$$

Solve for x_1:

$$x_1 = \frac{\text{s}}{1 + \frac{g_1}{g_2} + \frac{g_1}{g_3} + \cdots + \frac{g_1}{g_{k-1}}}$$

Because of Theorem 1, our total yield is defined in terms of p_k, and y_k is now defined as $g_1 x_1$,and x_1 is now redefined, it follows that:

$$\begin{aligned} Yld_k &= p_2 y_2 + p_3 y_3 + \cdots + p_n y_n \\ &= p_k y_k \\ &= p_k g_1 x_1 \\ &= \frac{p_k s}{\frac{1}{g_2} + \frac{1}{g_3} + \cdots + \frac{1}{g_{k-1}}} \end{aligned}$$

1.8 Theorem 2: The optimal sustainable yield is the largest value of Yld$_k$.

$$Yld_k = \frac{p_k s}{\frac{1}{g_2} + \frac{1}{g_3} + \cdots + \frac{1}{g_{k-1}}} \text{ for } k = 2, 3, \ldots, n$$

Remember that the corresponding value of k is the number of the class that is to be completely harvested. To find the Optimal Sustainable Yield each height class (Yld$_2$, Yld$_3$,...,Yld$_n$) must be solved for. The highest number resulting from the totals of all the height classes will indicate the class that should be completely harvested

1.8.1 Example

$$\mathbf{p} = \begin{bmatrix} 0 \\ 50 \\ 100 \\ 150 \\ 200 \\ 250 \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} .28 \\ .31 \\ .25 \\ .23 \\ .37 \\ 0 \end{bmatrix}, \quad G = \begin{bmatrix} .72 & 0 & 0 & 0 & 0 & 0 \\ .28 & .69 & 0 & 0 & 0 & 0 \\ 0 & .31 & .75 & 0 & 0 & 0 \\ 0 & 0 & .25 & .77 & 0 & 0 \\ 0 & 0 & 0 & .23 & .63 & 0 \\ 0 & 0 & 0 & 0 & .37 & 1 \end{bmatrix}$$

$$
\begin{aligned}
Yld_2 &= \frac{50s}{.28^{-1}} = 14.0s \\
Yld_3 &= \frac{100s}{.28^{-1} + .31^{-1}} = 14.7s \\
Yld_4 &= \frac{150s}{.28^{-1} + .31^{-1} + .25^{-1}} = 13.9s \\
Yld_5 &= \frac{200s}{.28^{-1} + .31^{-1} + .25^{-1} + .23^{-1}} = 13.2s \\
Yld_6 &= \frac{250s}{.28^{-1} + .31^{-1} + .25^{-1} + .23^{-1} + .37^{-1}} = 14.0s
\end{aligned}
$$

This example says that Yld_3 should be completely harvested and that the yield equals $14.7 times the total number of trees in the forest. The third class has neither the most amount of trees or commands the highest price. As a matter of fact, it would be the fifth class that would bring in the most amount of money. What this tells us is that the third class is the class that will bring in the most amount of money while keeping the existence of the forest intact

MatLab can be effectively used to find the Sustainable Harvesting Condition and the Optimal Sustainable Yield. The following m-files were written by M. Apple and D. Helliwell. The *forest* m-file gave specific values for the nonharvest vector **x**, the growth matrix, which we used to solve for the harvest vector **y**. The *example* m-file gave specific values for the price vector **p**. Using the same growth values, and the price values we applied theorem 2 and got the optimal sustainable yield.