The Fourier Series and the Discrete Fourier Transform

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Abstract: The Fourier Series and its applications to the Discrete Fourier Transform are discussed The paper is written in a colloquial style to avoid intimidating readers who are of a lesser level of intelligence and ingenuity as the vastly brilliant authors.

Introduction

Deemed one of the crowning achievements of the 20th Century, the Fourier Series has applications that are far reaching in various fields of science and mathematics. The Discrete Fourier Transform is one particular tool widely used in today's age of computers and solid state electronics. From graphic equalizers in stereos to the most advanced scientific sampling software, the usefulness of this mathematical feet is astounding.

Linear Algebra

A key idea in Linear Algebra is that a vector can be any abstract object. Take two arbitrary vectors v and w:

$$v = (v_1, v_2, \ldots, v_n)$$
$$w = (w_1, w_2, \ldots, w_n)$$

The dot product of two vectors is then the sum of the products of the individual elements making up the two vectors.

$$v \bullet w = (v_1 w_1 + v_2 w_2 + \ldots + v_n w_n)$$

Central to the Fourier Series (and consequently the Discrete Fourier Transform) is the fact that continuous functions can be thought of as vectors. Take two sinusoidal functions f and g:

$$f(x) = \sin(x)$$
$$g(x) = \cos(x)$$

When the dot product of these two continuous function "vectors" is taken, it becomes an inner product with infinitely many terms...thus an integral.

$$f \bullet g = \; < f\,,\, g > \; = \int_0^{2\pi} \sin(x)\cos(x)dx$$

The integrals of certain trigonometric functions (evaluated from zero to twice pi, the period of sinusoidal functions) turn out to be zero. (their inner products are zero).

$$< \sin(mx), \cos(nx) > \; = \int_0^{2\pi} \sin(mx)\cos(nx)dx = 0\,, \text{ for any integer m and n}$$
$$< \sin(mx), \sin(nx) > \; = \int_0^{2\pi} \sin(mx)\sin(nx)dx = 0\,,\ m \neq n$$
$$< \cos(mx), \cos(nx) > \; = \int_0^{2\pi} \cos(mx)\cos(nx)dx = 0\,,\ m \neq n$$

Thus by the omnipotent laws of Linear Algebra, the function “vectors” are orthogonal and can span a space of periodic functions. This means that any element in the function “space” can be written as a linear combination of these basis vector functions.

The Fourier Series

Now that we have established the concept of a set of functions that are orthogonal to one another we are ready to approach the Fourier Series. It is a fundamental concept in linear algebra that a given set of orthogonal vectors span a space that has a dimension equal to the number of vectors in the set. For instance, if we have three mutually orthogonal vectors we can think of them as spanning Cartesian 3-space. From this follows the concept that any object in Cartesian 3-space can be described as a linear combination of these basis vectors. The coefficients of this linear combination are regarded as that objects coordinates. The set of orthogonal functions that compose the Fourier series can be thought of as spanning the space of periodic functions. Thus, analogous to the vectors that span 3-space, this means that a periodic function can be described as a linear combination of the Fourier basis functions.

$$\begin{aligned} f(t) &= a_0 + a_1\cos(\omega_0 t) + b_1\sin(\omega_0 t) + a_2\cos(2\omega_0 t) + b_2\sin(2\omega_0 t) + ... \\ &= a_0 + \Sigma_{n=1}^{\infty}(a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)) \end{aligned}$$

where f(t) is a periodic function and $\omega_0 = \frac{2\pi}{T}$ is the fundamental angular frequency of the function (more on this later).

It should be noted that not all periodic functions are included in this span. A periodic function must satisfy three criteria known as the Dirichlet conditions. These conditions are as follows:

- f(t) is piecewise continuous.
- f(t) has isolated maxima and minima.
- f(t) is absolutely integrable over a period.

The challenge of the Fourier series now becomes to find the coefficients. This is where the concept of orthogonality really becomes our friend! If we wish to find the coefficient a_0 we take the inner product of each side of the equation with 1 (1 is the orthogonal function associated with a_0). We will then utilize the distributive property of the inner produce to eliminate all of the terms on the right side of the equation except the one including the desired coefficient.

$$\begin{aligned} < 1, f(t) > &= < 1, a_0 + a_1\cos(\omega_0 t) + b_1\sin(\omega_0 t) + ... > \\ < f(t), 1 > &= < 1, a_0 > + < 1, a_1\cos(\omega_0 t) > + < 1, b_1\sin(\omega_0 t) > + ... \\ \int_0^T f(t)dt &= \int_0^T a_0 dt + \int_0^T a_1\cos(\omega_0 t)dt + \int_0^T b_1\sin(\omega_0 t)dt > + ... \\ \int_0^T f(t)dt &= a_0 T + 0 + 0 + ... \\ a_0 &= \frac{\int_0^T f(t)dt}{T} \end{aligned}$$

Thus we have a general formula for finding the constant term, commonly called the D.C. coefficient. Now let’s take a general approach to finding the a_k coefficients.

$$< \cos(k\omega_0 t), f(t) > \; = \; < \cos(k\omega_0 t), a_0 + a_1\cos(\omega_0 t) + b_1\sin(\omega_0 t) + ... + a_k\cos(k\omega_0 t) + ...$$

$$\int_0^T f(t)\cos(k\omega_0 t)dt = \int_0^T a_0\cos(k\omega_0 t)dt + \int_0^T a_1\cos(k\omega_0 t)\cos(\omega_0 t)dt + \int_0^T b_1\cos(k\omega_0 t)\sin(\omega_0 t)dt + ...$$

$$... + \int_0^T a_k\cos^2(k\omega_0 t)dt + ...$$

$$\int_0^T f(t)\cos(k\omega_0 t)dt = 0 + 0 + 0 + ...a_k\int_0^T \cos^2(k\omega_0 t)dt$$

We will do a little side calculation to evaluate the right most integral. . .

$$\begin{aligned}\int_0^T \cos^2(k\omega_0 t)dt &= \int_0^T \frac{1}{2} + \frac{\cos(k\omega_0 t)}{2}dt \\ &= \frac{1}{2}[t + \frac{\sin(k\omega_0 t)}{k\omega_0}]_0^{T=\frac{2\pi}{\omega_0}} \\ &= \frac{T}{2}\end{aligned}$$

And back to our original goal . . .

$$\begin{aligned}\int_0^T f(t)\cos(k\omega_0 t)dt &= 0 + 0 + 0 + ...a_k\frac{T}{2} \\ a_k &= \frac{2}{T}\int_0^T f(t)\cos(k\omega_0 t)dt\end{aligned}$$

We now have a general way to find the a_0 and the a_k coefficients. To complete the picture we will perform the same trickery on for the b_k coefficients.

$$< \sin(k\omega_0 t), f(t) > \; = \; < \sin(k\omega_0 t), a_0 + a_1\cos(\omega_0 t) + b_1\sin(\omega_0 t) + ... + a_k\cos(k\omega_0 t) + ... >$$

$$\int_0^T f(t)\sin(k\omega_0 t)dt = \int_0^T a_0\sin(k\omega_0 t)dt + \int_0^T a_1\sin(k\omega_0 t)\cos(\omega_0 t)dt + \int_0^T b_1\sin(k\omega_0 t)\sin(\omega_0 t)dt + ...$$

$$... + \int_0^T b_k\sin^2(k\omega_0 t)dt + ...$$

$$\int_0^T f(t)\sin(k\omega_0 t)dt = 0 + 0 + 0 + ... + b_k\int_0^T \sin^2(k\omega_0 t)dt + ...$$

$$\int_0^T f(t)\sin(k\omega_0 t)dt = b_k\frac{T}{2}$$

$$b_k = \frac{2}{T}\int_0^T f(t)\sin(k\omega_0 t)dt$$

In order to put the concept in the proper perspective it only seems appropriate to walk through an example of how this series approximates a specific function. Let’s consider a function that has practical applications in cathode ray tubes known as the “sawtooth” wave.

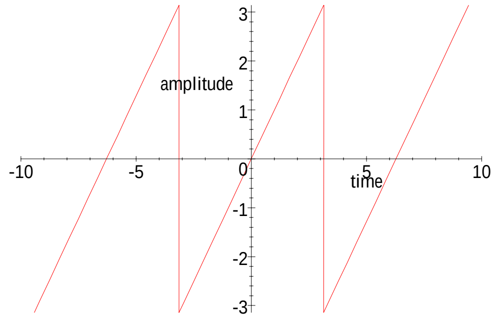

Figure 1. The “sawtooth” wave.

We need to note three things about this graph. First, the period is 2π which leads to $\omega_0 = \frac{2\pi}{2\pi} = 1$. Second, the period begins at $-\pi$ and ends at π. Third, and most importantly, the equation for this function is $f(t) = t$ for $t = [-\pi, \pi]$.

Let us find the Fourier coefficients for this function.

$$a_0 = \frac{1}{T}\int_0^T f(t)dt$$
$$= \frac{\int_{-\pi}^{\pi} tdt}{2\pi}$$
$$= \frac{1}{4\pi}[t^2]_{-\pi}^{\pi}$$
$$= 0$$

There will be no $a_0$coefficients.

$$a_k = \frac{2}{T}\int_{T_i}^{T_f} f(t)\cos(k\omega_0 t)dt$$
$$= \frac{2\int_{-\pi}^{\pi} t\cos(kt)dt}{2\pi}$$

t	$\cos(kt)$
1	$\frac{\sin(kt)}{k}$
0	$\frac{\cos(kt)}{k^2}$

$$= \frac{1}{\pi k^2}[tk\sin(kt) + \cos(kt)]_{-\pi}^{\pi}$$
$$= 0$$

Nor will there be any cosine terms.

$$
\begin{aligned}
b_k &= \frac{2}{T}\int_{T_i}^{T_f} f(t)\sin(k\omega_0 t)dt \\
&= \frac{2\int_{-\pi}^{\pi} t\sin(kt)dt}{2\pi} \\
& \begin{array}{ll} t & \sin(kt) \\ 1 & -\frac{\cos(kt)}{k} \\ 0 & -\frac{\sin(kt)}{k^2} \end{array} \\
&= \frac{1}{\pi k^2}[tk\cos(kt) + \sin(kt)]_{-\pi}^{\pi} \\
&= \frac{2\pi k\cos(k\pi)}{\pi k} \\
&= \frac{2(-1)^{n+1}}{n}
\end{aligned}
$$

We can now represent the sawtooth function as a series of sine functions with the b_k coefficients as derived above.

$$
\begin{aligned}
f(t) &= \Sigma_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin(nt) \\
&= 2\sin(t) - \sin(2t) + \frac{2}{3}\sin(3t) - \frac{1}{2}\sin(4t) + \frac{2}{5}\sin(5t)...
\end{aligned}
$$

Of course, in real applications we cannot carry the series out an infinite number of terms. We must limit the number of terms to a reasonable approximation of the original function. The following series of plots shows the progression from the second to the sixth iteration and the last shows 100 iterations.

The approximation improves with each iteration and the plots are nearly indistinguishable by n=100. The next plot shows each of the sinusoidal functions that sum up improves the 5th iteration.

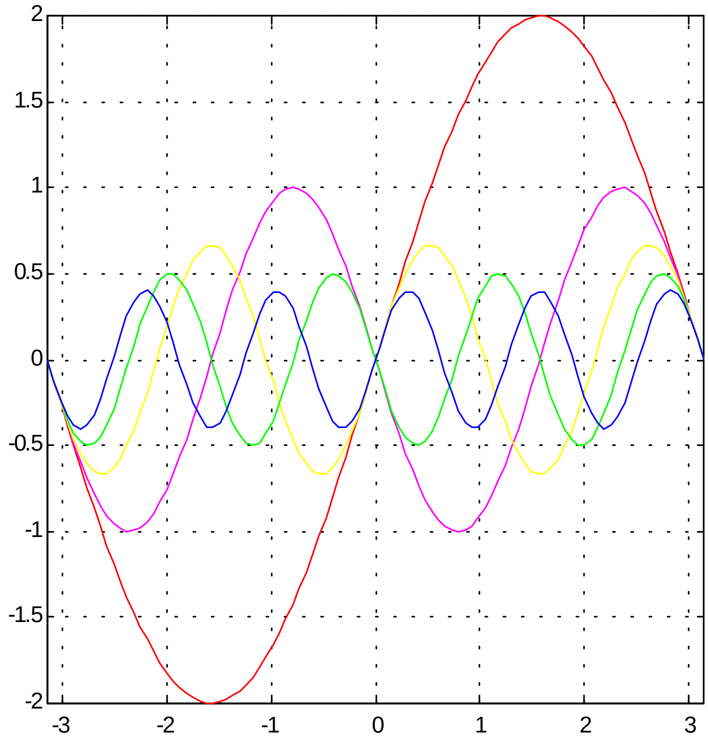

Figure 2. The superposition of the first five iterations.

A careful analysis of this plot will show that the frequency of the five plots on these axes are integer multiples of the frequency of the original function. This is not a mere coincidence, this is exactly what the Fourier series does. It takes a periodic function of a given frequency ($f = \frac{1}{T}$) and expresses it as a sum of sinusoids that are integer multiples of that frequency. The frequency of the original function is regarded as the fundamental frequency. This forms the cornerstone concept of the Discrete Fourier Transform.

As a matter of convention, the Fourier series is often depicted in its exponential form rather than the trigonometric form outlined up until now. If we consider the coefficients a_k and b_k as coordinate pairs in the plane then we can also think of them as cosine and sine coordinates in relation to an angle theta.

$$
\begin{aligned}
a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) &= \cos(\theta_k)\cos(k\omega_0 t) + \sin(\theta_k)\sin(k\omega_0 t) \\
&= \cos(k\omega_0 t + \theta_k)
\end{aligned}
$$

To make the final transition you will need to recall this calculus II identity.

$$\cos(kt + \theta_k) = \frac{e^{i(k\omega_0 t+\theta_k)} + e^{-i(k\omega_0 t+\theta_k)}}{2}$$
$$= \frac{1}{2}e^{i\theta_k}e^{ik\omega_0 t} + \frac{1}{2}e^{-i\theta_k}e^{-ik\omega_0 t}$$
$$\text{where; } X_k = \frac{1}{2}e^{i\theta_k}, \text{ and } X_{-k} = \frac{1}{2}e^{-i\theta_k}$$
$$\text{thus, } f(t) = \Sigma_{k=-\infty}^{k=\infty} e^{ik\omega_0 t}$$

As you can see this introduces complex numbers to the series. It becomes a new matter to interpret the series as a depiction of real phenomena. Regardless, this form is clearly more compact and is regarded as the most elegant form of the Fourier series.

The Discrete Fourier Transform

At this point one could either regard the Fourier series as a powerful tool or simply a mathematical contrivance. It may seem rather silly to rewrite a known function as a set of sinusoids simply to show that it can be done. The fact is up until now we have only been developing the theory for the real power of the Fourier series which is the Discrete Fourier Transform.

Periodic phenomena occur everywhere in the physical world. Every thing that we see can be broken down to a complex web of wavelengths. Every thing that we hear is transmitted to our brain by our ear's cochlea as a spectrum of frequencies. Nearly every type of communication is achieved my manipulating some sort of periodic phenomena. For this reason it becomes advantageous for us to have a mastery over the mathematics that govern these phenomena. The only problem is nature rarely delivers an equation to accompany its complexities.

As is the case with most of life's difficult situations we now resort to breaking the problem into a series of smaller steps. Let's say that we have some sort of complex sound and we want to determine each of the sound wave frequencies that it is composed of. The first thing we would want to do is compile some sort of data that could represent this sound numerically. In the case of sound, this would like lead us to some sort of recording device. The first thing that we will do is press the record button and let the data pour in until we press the stop button. Let's say that we record for 4 seconds. As mundane as this sounds so far this is actually an extremely important aspect of the Fourier Transform. Due to the limitations of time and our machinery we are only able to interpret phenomena over a finite quantity of time, in our case four seconds. Thus, by starting and stopping our recording we have done what is known as "windowing" our data.

However, we have not broken things down into small enough steps quite yet. Although, we are now merely dealing with a finite chunk of reality we still have a complex sound with no known mathematical description. What we can now do is measure the intensity of our sound at equal intervals of time. This is known as sampling. First we will pick the interval of time that will pass between each measurement. We will want that interval to be as short as possible thus our main limitation probably be our equipment. Let's say that we can take a measurement every .1 seconds. Thus, we can take 10 samples per second which is referred to as our sampling rate. This will give us a total of 40 samples of the sound phenomena

Now that we have windowed and sampled our data we can apply the Fourier magic to discover its constituent frequencies. The equation changes a little when we deal with discrete samples rather than a continuous function.

$$x(n\Delta t) = \Delta f \Sigma_{n=0}^{N-1} X_k(k\Delta f) e^{i2\pi k\Delta f_n \Delta t}$$

Note that this adds the mathematical discrete time interval of the samples delta t and the frequency interval delta f. Also note that N represents the total number of samples taken. This is actually known as the "inverse DFT." The actual DFT equation returns a set of coefficients for a given sample value.

$$X(k\Delta f) = \Delta t \Sigma_{n=0}^{N-1} x(n\Delta t) e^{-i2\pi k\Delta f_n \Delta t}$$

These equations looks very intimidating but it is only a step farther than the last depiction.

Now let's return to our sound example. As a matter of convention we will want to find the same number of Fourier coefficients for each data sample as number of data samples we have. Thus, we will want to find the first 40 Fourier coefficients for each of these samples. This leads us to a 40 by 40 matrix with columns being time entries and rows being frequencies. If we set out to compute each of these entries by hand it will not take long to realize that this requires a whole lot of pencil lead and even more patience! This matrix requires1600 individual calculations!

This leads us to implement the use of modern computers to perform these computations for us. However even with the amazingly high clock speeds of today's Pentium machines, calculating the DFT requires a substantial amount of time for all but the shortest windows of data. This fact led to the development of the Fast Fourier Transform (FFT). Although the details of the FFT algorithm are beyond the scope of this article it should be noted that the essential mathematics remains unchanged. It merely performs the computations extremely quickly.

Conclusion

Without the powerful mathematics behind the Fourier Series and more importantly the Fourier Transform, the 20th Century would be as destitute as the 8th. The mere simple task of tuning your Rob-E-BASS cd player to reverberate successfully throughout your car cabin, or viewing "I Know What You Did Last Summer" on high definition T.V. would be impossible. Science would not have the privilege of high-tech wave-sampling software. We would all be subject to darkness, pestilence, terror and plague. It is no understatement that the DFT is considered one of the crowning achievements of the 20th Century. It has been an honor and a privilege to bring it to you, the reader, in this user friendly, low fat and no cholesterol form.

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