GPS in Linear Algebra: The N-Site Problem

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Abstract

This article explicates the mathematical process of determining the location of a known object of interest by using GPS from N number of passive sensors.

Introduction

The global positioning system (GPS) is used very commonly in the world of navigation today. Whether they occur in automobiles or airplanes, GPS units are all depended upon for exceptional accuracy–some obviously more so than others. Receivers on earth can determine positions only by the information they collect from a number of satellites orbiting the earth. These satellites are arranged on six orbital planes–four on each plane, totalling twenty-four. This configuration places an orbital plane at every sixty degrees surrounding the Earth, maximizing exposure to terrestrial receivers. A traditional receiver collects information from as many of these satellites as possible, then determines location using the position of each satellite, and the time it takes a radio signal to travel from the satellite to the receiver. This type of receiver is known as an active receiver. What then, would result in using a device that is inactive, or passive? This would mean that there is no constant information shared between satellite and receiver. The point of this project is to analyze the accuracy that can be achieved by using passive equipment, which is known as the N-Site problem.

The N-Site Proposal

The problem presented by using N passive sensors creates an ideal opportunity for triangulation. The situation consists of, for the purposes of simplicity, three sensors that are pointed toward an object of interest. These sensors have positions that are known, thanks to information received from satellites. Each sensor is pointed in a direction that seems to be toward the object of interest. Knowing this, it suffices to say that in a perfect situation, the line of sight of each sensor toward the object should intersect at the object. This, however, is not the case because there are many aspects that cause error in the directions generated by each sensor. Some contributing factors to error include atmospheric refraction, untrue sensor alignment, and poor system conditions. Error in these directions inevitably cause the lines of sight to miss instead of intersect at a common point. This creates an area, bound by the lines of sight, that encompass the object of interest. The objective, then, is to come up with a determination of the location of the object. This situation leads us to use triangulation.

Geometric Representation

To represent this situation, we will refer to a two dimensional model (Figure 1) containing three sites of passive sensors $\mathbf{s}_1$, $\mathbf{s}_2$, and $\mathbf{s}_3$. The concept of this problem allows N passive sensors, that in reality occur in three dimensional space. Simplification of this problem is necessary and also sufficient in illustrating the general process executed. The lines of sight of each sensor extend out from each sensor in the direction toward the object of interest. Since they do not intersect at a single point, they extend to A_1, A_2, and A_3, respectively. These lines of sight encompass an area (triangle) in which the object is located. This triangle, bound by A_1, A_2, and A_3, has a point in it $\mathbf{r}$ (object of interest) where three lines originating from I, J, and K meet. Each line originating from I, J, and K are perpendicular to the lines of sight of the passive sensors, representing the shortest distance to a point at which the lines of sight can meet. Knowing this, we can square these distances, add them up, take the gradient to get the mean square error, and then set it equal to zero to find the value that will minimize this error. This is the process that will be taken in solving the N-Site problem.

Before we dive into this process, however, we must establish some additional vectors. To do this, let us refer to Figure 2, which is a close up of the situation that occurs at one sensor. The vector $\mathbf{s}_i$ is a vector extending from the origin to a sensor site, showing how a sensor can be located anywhere in space. The line of sight of this vector toward the object of interest is given by the dashed blue line. The true path from the sensor to the object is given by the red vector $\mathbf{r}_i$. We proceed by projecting the vector $\mathbf{r}_i$ onto the blue unit vector $\mathbf{d}_i$, which now lies on the line of sight of the sensor.

$$p_i\mathbf{d}_i = \frac{\mathbf{r}_i^T\mathbf{d}_i}{\mathbf{d}_i^T\mathbf{d}_i}\mathbf{d}_i = (\mathbf{r}_i^T\mathbf{d}_i)\mathbf{d}_i.$$

The term on the bottom, $\mathbf{d}_i^T\mathbf{d}_i$, is dropped since the dot product of a unit vector with itself is one. To get the error vector we must then subtract this projection from $\mathbf{r}_i$, which will give us

$$\mathbf{e}_i = \mathbf{r}_i - \frac{\mathbf{r}_i^T\mathbf{d}_i}{\mathbf{d}_i^T\mathbf{d}_i}\mathbf{d}_i = \mathbf{r}_i - (\mathbf{r}_i^T\mathbf{d}_i)\mathbf{d}_i.$$

To simplify, we can substitute p_i in for the signed length of the projection of $\mathbf{r}_i$ onto $\mathbf{d}_i$.

$$\mathbf{e}_i = \mathbf{r}_i - p_i\mathbf{d}_i \tag{1}$$

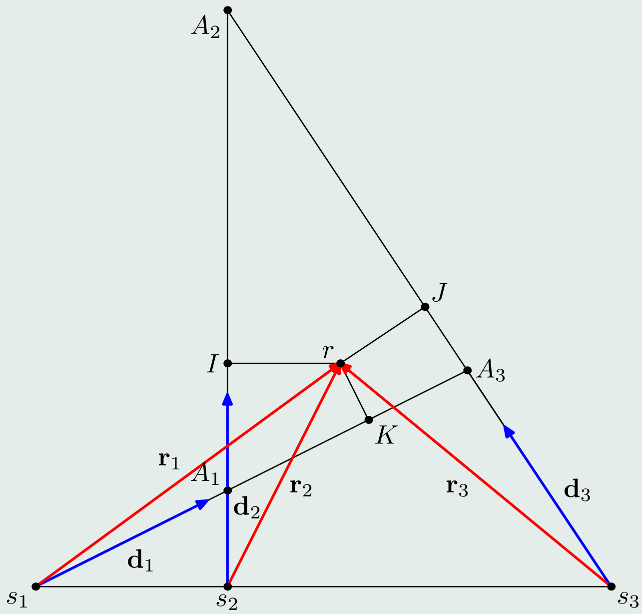

Figure 1: Geometric Representation.

The Simplified Mathematics Behind the Problem

Since the sensors are not perfect and do not intersect at one point, we must determine the perpendicular distances from each line of sight to the closest point where they could meet. Hypothetically, the best method for determining the location of a point of interest, is finding the smallest sum of the perpendicular distances between the lines of sight; these perpendicular lines, in theory, will intersect at the point of interest. In order to solve for this distance we must set up a coordinate system. This coordinate system will include both positive and negative values; to obtain correct distances we must either square the distances, or use absolute values. A well proven and accepted method for calculating the error which comes from these calculations is using Mean Square Error; this eliminates negative numbers and will allow us to accurately compare the distance values. To calculate the perpendicular distance, we use the error vector (1) previously calculated.

$$\mathbf{e}_i = \mathbf{r}_i - p_i \mathbf{d}_i$$

With this error vector known (green vector $\mathbf{e}_1$ in Figure 2), we can proceed by adding up all of the squares of this vector, as it exists with each sensor. We can express the mean-square-error as E^2.

$$E^2 = \frac{1}{N} \sum_{i=1}^{N} \|\mathbf{r}_i - p_i \mathbf{d}_i\|^2 \tag{2}$$

Since a vector quantity squared equals the vectors transpose times the vector, $||a||^2 = a^T a$, we can rewrite equation (2) as follows.

$$E^2 = \frac{1}{N} \sum_{i=1}^{N} (\mathbf{r}_i - p_i \mathbf{d}_i)^T (\mathbf{r}_i - p_i \mathbf{d}_i).$$

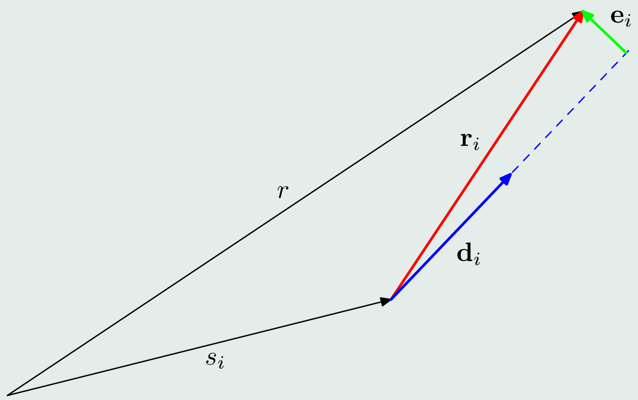

Figure 2: Independent Passive Sensor Close-Up.

$$E^2 = \frac{1}{N}\sum_{i=1}^{N}(\mathbf{r}_i^T - p_i\mathbf{d}_i^T)(\mathbf{r_i} - p_i\mathbf{d}_i) \tag{3}$$

Now by expanding equation (3) and combining like terms, we get as follows.

$$E^2 = \frac{1}{N}\sum_{i=1}^{N}\mathbf{r}_i^T\mathbf{r}_i - 2p_i\mathbf{r}_i^T\mathbf{d}_i + p_i^2\mathbf{d}_i^T\mathbf{d}_i. \tag{4}$$

From the projection matrix we have $\mathbf{d}_i^T\mathbf{d}_i = 1$ and $p_i = \mathbf{r}_i^T\mathbf{d}_i$, so we can use these relationships to rewrite equation (4) as follows.

$$\begin{aligned} E^2 &= \frac{1}{N}\sum_{i=1}^{N} \mathbf{r}_i^T\mathbf{r}_i - 2p_i^2 + p_i^2 \\ &= \frac{1}{N}\sum_{i=1}^{N} \mathbf{r}_i^T\mathbf{r}_i - p_i^2. \end{aligned}$$

From Figure 2, one can see that $\mathbf{r}_i = \mathbf{r} - \mathbf{s}_i$, so

$$\begin{aligned} E^2 &= \frac{1}{N}\sum_{i=1}^{N}(\mathbf{r}-\mathbf{s}_i)^T(\mathbf{r}-\mathbf{s}_i) - (\mathbf{r}_i^T\mathbf{d}_i)^2 \\ &= \frac{1}{N}\sum_{i=1}^{N}(\mathbf{r}-\mathbf{s}_i)^T(\mathbf{r}-\mathbf{s}_i) - (\mathbf{r}-\mathbf{s}_i)^T\mathbf{d}_i(\mathbf{r}-\mathbf{s}_i)^T\mathbf{d}_i \\ &= \frac{1}{N}\sum_{i=1}^{N}(\mathbf{r}-\mathbf{s}_i)^T(\mathbf{r}-\mathbf{s}_i) - (\mathbf{r}^T\mathbf{d}_i - \mathbf{s}_i^T\mathbf{d}_i)(\mathbf{r}^T\mathbf{d}_i - \mathbf{s}_i^T\mathbf{d}_i) \\ &= \frac{1}{N}\sum_{i=1}^{N}\mathbf{r}^T\mathbf{r} - 2\mathbf{r}^T\mathbf{s}_i + \mathbf{s}_i^T\mathbf{s}_i - (\mathbf{r}^T\mathbf{d}_i)^2 + 2\mathbf{r}^T\mathbf{d}_i\mathbf{s}_i^T\mathbf{d}_i - (\mathbf{s}_i^T\mathbf{d}_i)^2. \qquad (5) \end{aligned}$$

In order to find the minimum error we take the gradient of (5). $\mathbf{s}_i$ and $\mathbf{d}_i$ are constant vectors, and $\mathbf{r}$ is an unknown vector which represents the coordinates of the object as follows.

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

We can cancel out all of the terms not containing an $\mathbf{r}$ in equation (5). These canceled terms are $\mathbf{s}_i^T\mathbf{s}_i$ and $(\mathbf{s}_i^T\mathbf{d}_i)^2$. The equation now becomes

$$\nabla E^2 = \frac{1}{N}\sum_{i=1}^{N}\nabla(\mathbf{r}^T\mathbf{r} - 2\mathbf{r}^T\mathbf{s}_i - (\mathbf{r}^T\mathbf{d}_i)^2 + 2\mathbf{r}^T\mathbf{d}_i\mathbf{s}_i^T\mathbf{d}_i). \tag{6}$$

We can now take the gradient of each piece of equation (6), starting with $\mathbf{r}^T\mathbf{r}$.[1]

$$\begin{aligned}\nabla_r\mathbf{r}^T\mathbf{r} &= \nabla_r\|\mathbf{r}\|^2\\ &= \nabla_r r^2\\ &= 2r\nabla_r r\\ &= 2r\cdot\frac{\mathbf{r}}{r}\\ &= 2\mathbf{r}\end{aligned}$$

Next we take the gradient of $-2\mathbf{r}^T\mathbf{s}_i$

$$\begin{aligned}-\nabla 2\mathbf{r}^T\mathbf{s}_i &= -2\nabla\begin{bmatrix}r_1 & r_2 & r_3\end{bmatrix}\cdot\begin{bmatrix}s_1\\ s_2\\ s_3\end{bmatrix}\\ &= -2\nabla(r_1s_1 + r_2s_2 + r_3s_3)\\ &= -2\langle s_1, s_2, s_3\rangle\\ &= -2\mathbf{s}_i\end{aligned}$$

Now taking the gradient of $-(\mathbf{r}^T\mathbf{d}_i)^2$ yields the following term.

$$\begin{aligned}-\nabla(\mathbf{r}^T\mathbf{d}_i)^2 &= -2(\mathbf{r}^T\mathbf{d}_i)^1\nabla\mathbf{r}^T\mathbf{d}_i\\ &= -2(\mathbf{r}^T\mathbf{d}_i)\mathbf{d}_i\end{aligned}$$

[1] See appendix for substitution of $\nabla_r r$ for $\frac{\mathbf{r}}{r}$.

The last term we take the gradient of is $-2\mathbf{r}^T\mathbf{d}_i\mathbf{s}_i^T\mathbf{d}_i$

$$\begin{aligned} -\nabla 2\mathbf{r}^T\mathbf{d}_i\mathbf{s}_i^T\mathbf{d}_i &= -2\mathbf{s}_i^T\mathbf{d}_i\nabla\mathbf{r}^T\mathbf{d}_i \\ &= -2(\mathbf{s}_i^T\mathbf{d}_i)\mathbf{d}_i \end{aligned}$$

With the gradient of each part of equation (6) taken, we can set the gradient equal to zero, thus, minimizing the mean square error.

$$\nabla_r E^2 = \frac{1}{N}\sum_{i=1}^{N} 2\mathbf{r} - 2\mathbf{s}_i - 2(\mathbf{r}^T\mathbf{d}_i)\mathbf{d}_i + 2(\mathbf{s}_i^T\mathbf{d}_i)\mathbf{d}_i = 0 \tag{7}$$

Expression (7) contains two projection vectors, both onto the i^{th} line of sight. We now collect the terms that involve $\mathbf{r}$ and $\mathbf{s}_i$.

$$\frac{1}{N}\sum_{i=1}^{N} 2\mathbf{r} - 2(\mathbf{r}^T\mathbf{d}_i)\mathbf{d}_i - 2\mathbf{s}_i + 2(\mathbf{s}_i^T\mathbf{d}_i)\mathbf{d}_i = 0$$

$$\frac{1}{N}\sum_{i=1}^{N} 2\mathbf{r} - 2(\mathbf{r}^T\mathbf{d}_i)\mathbf{d}_i = \frac{1}{N}\sum_{i=1}^{N} 2\mathbf{s}_i - 2(\mathbf{s}_i^T\mathbf{d}_i)\mathbf{d}_i \tag{8}$$

Where $2\mathbf{r} - 2(\mathbf{r}^T\mathbf{d}_i)\mathbf{d}_i$ is the shortest vector from the object of interest to the projection of vector $\mathbf{r}$ onto the i^{th} line of sight that is translated to the origin; denoted in (9) as $\mathbf{r} - m_i\mathbf{d}_i$. $2\mathbf{s}_i - 2(\mathbf{s}_i^T\mathbf{d}_i)\mathbf{d}_i$ is the shortest vector extending from the projection of $\mathbf{s}_i$ onto the translated line of sight to the corresponding sensor; Figure 3 denotes this as $\mathbf{s}_i - n_i\mathbf{d}_i$. By multiplying equation (8) by $N/2$ and substituting in m_i and n_i, we can rewrite equation (8) as follows.

$$\sum_{i=1}^{N} \mathbf{r} - m_i\mathbf{d}_i = \sum_{i=1}^{N} \mathbf{s}_i - n_i\mathbf{d}_i \tag{9}$$

If all the lines-of-sight intersect at $\mathbf{r}$, one has the solution to the N-sight problem. However, in real life this is not the case, $\mathbf{r}$ is not on any of the line-of-sight vectors. There is always going to be a difference between $\mathbf{r} - m_i\mathbf{d}_i$ and $\mathbf{s}_i - n_i\mathbf{d}_i$, but this is allowed as long as the sum of the discrepancy over all of i is equal to zero. With passive sensors, there can be large errors when a site is located far from $\mathbf{r}$, because a small difference in angles between $\mathbf{r}_i$ and $\mathbf{d}_i$ can cause a huge difference between $\mathbf{r} - m_i\mathbf{d}_i$ and $\mathbf{s}_i - n_i\mathbf{d}_i$. Again, this can easily be seen when looking at Figure 3. When numbers are involved, it is convenient to express $\sum_{i=1}^{N} \mathbf{r} - m_i\mathbf{d}_i = \sum_{i=1}^{N} \mathbf{s}_i - n_i\mathbf{d}_i$ in a different form, namely $\mathbf{P} \cdot \mathbf{r} = \mathbf{u}$.

Since we substituted m_i for $\mathbf{r}^T\mathbf{d}_i$ in equation (9),

$$\begin{aligned} \mathbf{r} - m_i\mathbf{d}_i &= \mathbf{r} - (\mathbf{r}^T\mathbf{d}_i)\mathbf{d}_i \\ &= \mathbf{r} - \mathbf{d}_i(\mathbf{r}^T\mathbf{d}_i) \\ &= \mathbf{r} - \mathbf{d}_i(\mathbf{d}_i^T\mathbf{r}). \end{aligned} \tag{10}$$

In an effort to factor out $\mathbf{r}$, we must multiply equation (10) by an identity matrix, represented by I.

$$\begin{aligned} \mathbf{r} - m_i\mathbf{d}_i &= I\mathbf{r} - (\mathbf{d}_i\mathbf{d}_i^T)\mathbf{r} \\ &= (I - \mathbf{d}_i\mathbf{d}_i^T)\mathbf{r} \end{aligned} \tag{11}$$

With $\mathbf{r}$ factored out (11), we can now begin to write

$$\sum_{i=1}^{N} \mathbf{r} - m_i\mathbf{d}_i = \sum_{i=1}^{N} \mathbf{s}_i - n_i\mathbf{d}_i$$

in the form $\mathbf{P} \cdot \mathbf{r} = \mathbf{u}$. We have already established that $P\mathbf{r} = \mathbf{r} - m_i\mathbf{d}_i = (I - \mathbf{d}_i\mathbf{d}_i^T)\mathbf{r}$.

So

$$\left(\sum_{i=1}^{N}\left[I-\mathbf{d}_i\mathbf{d}_i^T\right]\right)\mathbf{r}=\sum_{i=1}^{N}\mathbf{s}_i-n_i\mathbf{d}_i.$$

This is in the form $\mathbf{P}\cdot\mathbf{r}=\mathbf{u}$ where

$$\mathbf{P}=\sum_{i=1}^{N}I-\mathbf{d}_i\mathbf{d}_i^T$$

$$\mathbf{u}=\sum_{i=1}^{N}\mathbf{s}_i-n_i\mathbf{d}_i.$$

Sample Problem

To give truth to the general math we have been deriving, we will now proceed with an example. The values of a two dimensional figure, such as Figure 1, will be used. The vectors representing the location of each sensor are

$$\mathbf{s}_1=\begin{bmatrix}-1\\0\end{bmatrix},\qquad \mathbf{s}_2=\begin{bmatrix}0\\0\end{bmatrix},\qquad \text{and}\qquad \mathbf{s}_3=\begin{bmatrix}2\\0\end{bmatrix}.$$

The unit vectors we will use, seen in Figure 1 as blue vectors $\mathbf{d}_1$, $\mathbf{d}_2$, and $\mathbf{d}_3$, are

$$\mathbf{d}_1=\frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\end{bmatrix},\qquad \mathbf{d}_2=\begin{bmatrix}0\\1\end{bmatrix},\qquad \text{and}\qquad \mathbf{d}_3=\frac{1}{\sqrt{13}}\begin{bmatrix}2\\1\end{bmatrix}$$

Using the relationship $P\cdot\mathbf{r}=\mathbf{u}$, we proceed with first defining P as follows.

$$P=\begin{bmatrix}3&0\\0&3\end{bmatrix}-\frac{1}{5}\begin{bmatrix}4&2\\2&1\end{bmatrix}-\begin{bmatrix}0&0\\0&1\end{bmatrix}-\frac{1}{13}\begin{bmatrix}4&-6\\-6&9\end{bmatrix}=\frac{1}{65}\begin{bmatrix}123&4\\4&72\end{bmatrix}$$

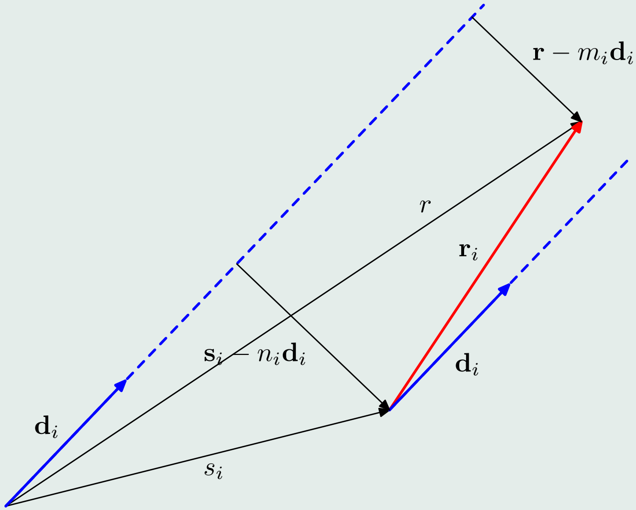

Figure 3: Derivation of $P \cdot \mathbf{r} = \mathbf{u}$.

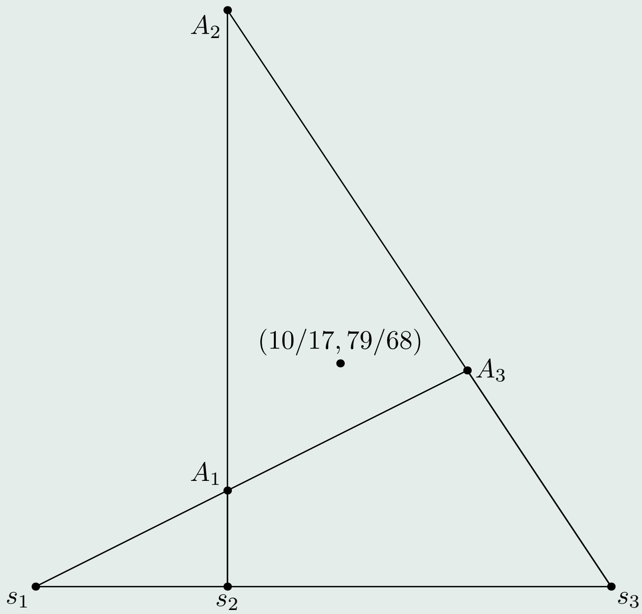

Figure 4: Object Location.

The vector $\mathbf{u}$ is found by first evaluating the n_i term for each sensor, resulting in $n_1 = -2/\sqrt{5}, n_2 = 0$, and $n_3 = -4/\sqrt{13}$. With these terms, we can now define $\mathbf{u}$ as

$$\mathbf{u} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \frac{2}{5} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} - 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \frac{4}{13} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \frac{1}{65} \begin{bmatrix} 77 \\ 86 \end{bmatrix}.$$

To compute $\mathbf{r}$, we can re-write our P and $\mathbf{u}$ using the expression

$$\mathbf{r} = P^{-1} \cdot \mathbf{u} = \frac{1}{136} \begin{bmatrix} 72 & -4 \\ -4 & 123 \end{bmatrix} \cdot \frac{1}{65} \begin{bmatrix} 77 \\ 86 \end{bmatrix} = \begin{bmatrix} 10/17 \\ 79/68 \end{bmatrix}.$$

Thus, we have the coordinates of $\mathbf{r}$, which is the location of the object of interest.

$$\mathbf{r} = \begin{bmatrix} 10/17 \\ 79/68 \end{bmatrix}.$$

This resulting coordinate fortunately falls in the triangle that is created by the line of sight vectors, encompassed by A_1, A_2, and A_3 in (Figure 3). In an effort to tie together the corresponding information available in Figures 1, 2, and 3, we can compare the error vectors. By using Figure 2, we can see that the error vector is

$$\mathbf{e}_i = \mathbf{r}_i - p_i \mathbf{d}_i.$$

This vector is seen in Figure 1 to represent the distance from I to $\mathbf{r}$, from J to $\mathbf{r}$, or from K to $\mathbf{r}$. The same distance can also be obtained by using Figure 3. In this picture, the error vector is the difference between the shortest distance from the i^{th} line of site translated to the origin to the sensor site, and the shortest distance from that same translated line of site to the object of interest.

$$\mathbf{e}_i = (\mathbf{s}_i - n_i \mathbf{d}_i) - (\mathbf{r} - m_i \mathbf{d}_i). \tag{12}$$

These two vectors (12) that represent the shortest distance between the translated line of sight and the object of interest and the sensor, should be equal. Having zero error, however, is not our situation when dealing with the N-Site Problem.

Conclusion

In this paper, we have explored the accuracy achieved by using passive GPS sensors to determine the location of an object of interest. We can safely conclude that the process executed in this work proved to be fairly accurate. The line of sight of each sensor miss each other and, as a result, form a triangle. This triangle represents an approximation of the location of the object of interest. Our accuracy, then, is supported by the fact that our calculated position of the object falls within this triangle.

Linear Algebra enabled this problem to be worked out in a simplified manner. The process included finding the distance of the error in each line of sight, which was found using projections onto unit vectors. These distances were then squared, added up, and exposed to the process of taking the gradient. The gradient is then set equal to zero in order to minimize this error. The resulting minimization gives the equation that is used to find the coordinate of the object of interest. This method of triangulation represents a basic form of procedure among the ideas utilized by the Global Positioning Systems of today.

Appendix

1. Let $\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$
\begin{aligned}
\nabla r &= \nabla \sqrt{x^2 + y^2 + z^2} \\
&= \left\langle \frac{\partial}{\partial x} \sqrt{x^2 + y^2 + z^2}, \frac{\partial}{\partial y} \sqrt{x^2 + y^2 + z^2}, \frac{\partial}{\partial z} \sqrt{x^2 + y^2 + z^2} \right\rangle \\
&= \left\langle \frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{1}{2}}(2x), \frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{1}{2}}(2y), \frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{1}{2}}(2z) \right\rangle \\
&= \left\langle \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right\rangle \\
&= \frac{1}{\sqrt{x^2 + y^2 + z^2}} \langle x, y, z \rangle \\
&= \frac{1}{r}\mathbf{r} \\
&= \frac{\mathbf{r}}{r}
\end{aligned}
$$

References

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[4] Thompson, Richard B. *Global Positioning System: The Mathematics of GPS Receivers.* Mathematics Magazine. vol. 71 no. 4 Oct. 1998 260-269.