Interpolating Data with the Discrete Fourier Transform

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Introduction

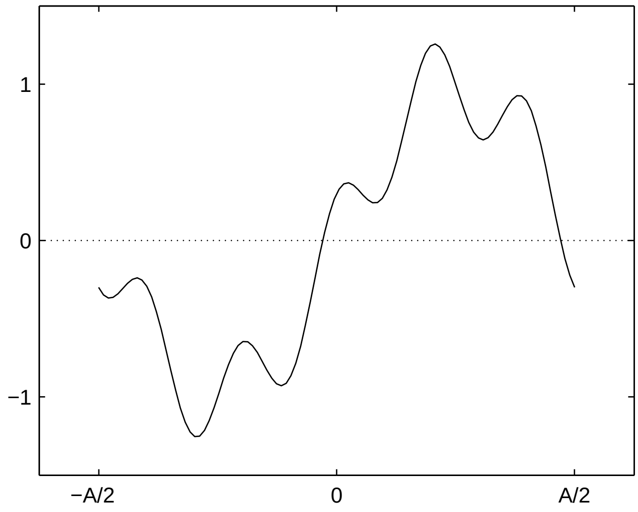
The Fourier transform says that any function can be approximated with an infinite series of sines and cosines. But often we need to approximate data that does not fit our conventional idea of a function. In this presentation we will derive the discrete Fourier transform, interpolate a set of data, and talk about a few applications of the DFT.

History

- 1754 Clairaut writes a paper containing the first ever DFT.
- 1805 Gauss uses an early version of the DFT to approximate orbits. Not discovered until 1865.
- 1807 Fourier's paper is rejected.
- 1812 Fourier wins Grand Prize at the Paris Academy.

Deriving the DFT

Given some periodic function:



- Defined on an interval A.
- Centered at the origin.
- $g(-A/2) = g(A/2)$

Fourier Transform

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi\omega x} \, dx, \tag{1}$$

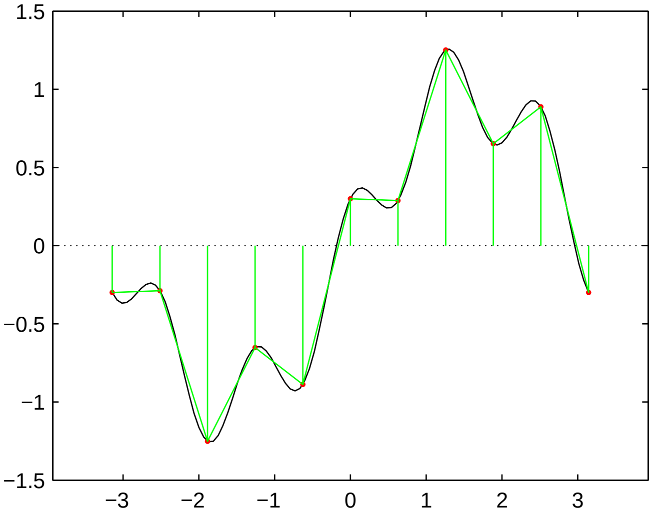
where

$$e^{\pm i2\pi\omega x} = \cos(2\pi\omega x) \pm i \sin(2\pi\omega x). \tag{2}$$

Rewritten as:

$$\hat{f}(\omega) = \int_{-\frac{A}{2}}^{\frac{A}{2}} f(x) e^{-i2\pi\omega x} \, dx. \tag{3}$$

Trapezoid Rule



Dividing the interval up:

- N equally spaced intervals of Δx.
- $A = N\Delta x$ or $\Delta x = A/N$

Using The trapezoid rule we can approximate this integral

$$\int_{-\frac{A}{2}}^{\frac{A}{2}} g(x)dx \approx \frac{\Delta x}{2}\left\{ g\left(-\frac{A}{2}\right) + 2\sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}-1} g(x_n) + g\left(\frac{A}{2}\right)\right\}. \quad (4)$$

Which becomes

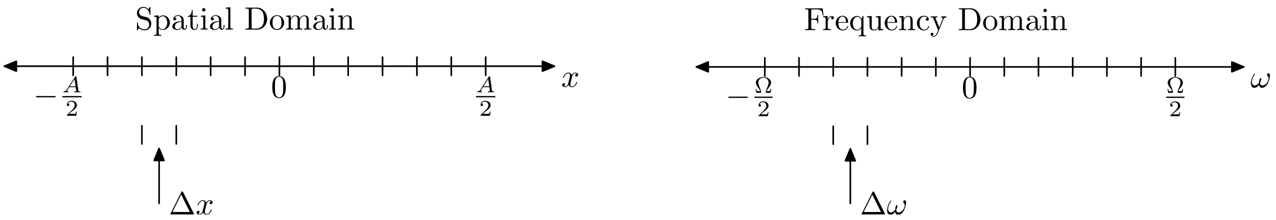
$$\int_{-\frac{A}{2}}^{\frac{A}{2}} g(x)dx \approx \Delta x \sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} g(x_n). \quad (5)$$

Substituting for Δx and our function

$$\hat{f} = \frac{A}{N} \sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} f(x_n)e^{-i2\pi\omega x_n}. \quad (6)$$

What frequencies do we use?

Reciprocity Relations



1. $A\Omega = N$.
2. $\Delta x \Delta\omega = \frac{1}{N}$.

Reciprocity relations allow us to replace the terms ωx_n

$$\hat{f} = \frac{A}{N} \sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} f(x_n) e^{-i2\pi\omega x_n}. \tag{7}$$

$$\omega x_n = \frac{nk}{N}. \tag{8}$$

giving us

$$\hat{f} = \frac{A}{N} \sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} f_n e^{-i2\pi nk/N}. \tag{9}$$

Our DFT coefficients are given by

$$F_k = \frac{1}{N} \sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} f_n e^{-i2\pi nk/N}. \tag{10}$$

Computing Coefficients

We want to compute the real and complex coefficients separately. We can do this using the formulas

$$Re\{F_k\} = \frac{1}{N} \sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} f_n \cos\left(\frac{2\pi nk}{N}\right). \tag{11}$$

Similarly, we can compute the complex coefficients using the formula

$$Im\{F_k\} = \frac{1}{N} \sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} -f_n \sin\left(\frac{2\pi nk}{N}\right). \tag{12}$$

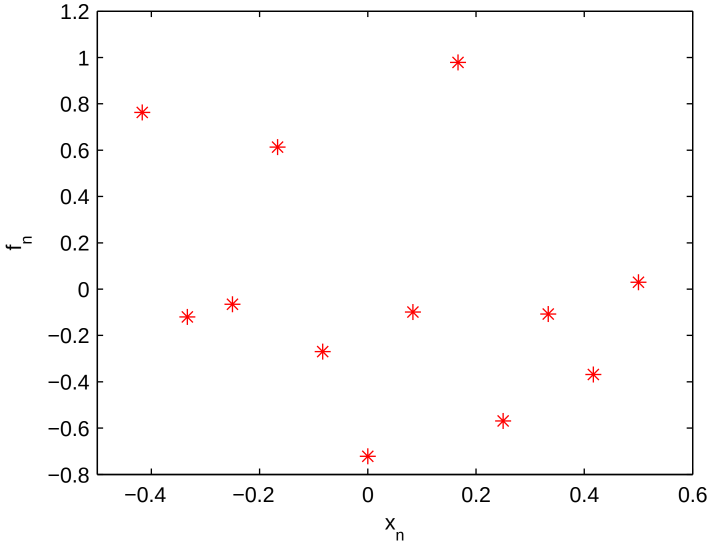
Let's Do It

Given the twelve equally spaced data points,

n, k	x_n	$Re\{f(x_n)\}$
-5	$-5/12$	0.7630
-4	$-4/12$	-0.1205
-3	$-3/12$	-0.0649
-2	$-2/12$	0.6133
-1	$-1/12$	-0.2697
0	0	-0.7216
1	$1/12$	-0.0993
2	$2/12$	0.9787
3	$3/12$	-0.5689
4	$4/12$	-0.1080
5	$5/12$	-0.3685
6	$6/12$	0.0293

(13)

find a trigonometric function that passes through all twelve points.



Substituting $N = 12$ into the summation, gives us

$$Re\{F_k\} = \frac{1}{12}\sum_{n=-5}^{6} f_n \cos\left(\frac{2\pi nk}{12}\right). \tag{14}$$

Similarly, we can calculate the imaginary coefficients by substituting

$N = 12$, giving us

$$Im\{F_k\} = \frac{1}{12} \sum_{n=-5}^{6} -f_n \sin\left(\frac{2\pi nk}{12}\right). \quad (15)$$

n, k	x_n	$Re\{f(x_n)\}$	$Re\{F_k\}$	$Im\{F_k\}$
-5	$-5/12$	0.7630	0.0684	-0.1093
-4	$-4/12$	-0.1205	-0.1684	0.0685
-3	$-3/12$	-0.0649	-0.2143	-0.0381
-2	$-2/12$	0.6133	-0.0606	0.1194
-1	$-1/12$	-0.2697	-0.0418	-0.0548
0	0	-0.7216	0.0052	0
1	$1/12$	-0.0993	-0.0418	0.0548
2	$2/12$	0.9787	-0.0606	-0.1194
3	$3/12$	-0.5689	-0.2143	0.0381
4	$4/12$	-0.1080	-0.1684	-0.0685
5	$5/12$	-0.3685	0.0684	0.1093
6	$6/12$	0.0293	0.1066	0

(16)

- $k = 0$ is analogous to your initial condition and has no associated frequency.
- $k = \pm 1$ has an associated frequency $\cos(\pi n/6) \mp i \sin(\pi n/6)$.
- $k = \pm 2$ has an associated frequency $\cos(\pi n/3) \mp i \sin(\pi n/3)$.
- $k = \pm 3$ has an associated frequency $\cos(\pi n/2) \mp i \sin(\pi n/2)$.
- $k = \pm 4$ has an associated frequency $\cos(2\pi n/3) \mp i \sin(2\pi n/3)$.
- $k = \pm 5$ has an associated frequency $\cos(5\pi n/6) \mp i \sin(5\pi n/6)$.
- $k = 6$ has an associated frequency $\cos(\pi n)$ or $(-1)^n$.

Creating an Interpolating Function

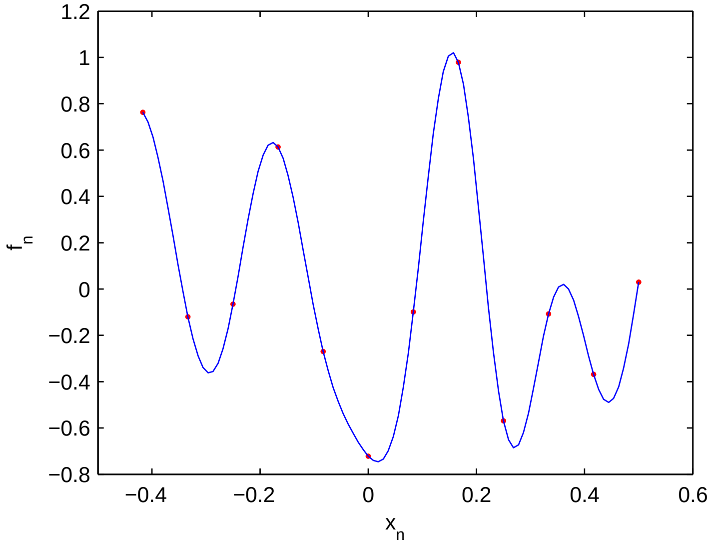
We can create our interpolating function with the inverse discrete Fourier transform which is defined as

$$f(x) = \sum_{k=-N/2+1}^{N/2} F_k e^{i2\pi kx/N}. \tag{17}$$

Which can be rewritten as

$$f(x) = Re\{F_0\} + 2\sum_{k=1}^{5}\left(Re\{F_k\}\cos\left(\frac{2\pi kx}{N}\right) - Im\{F_k\}\sin\left(\frac{2\pi kx}{N}\right)\right)$$
$$+ Re\{F_6\}\cos\left(\frac{2\pi kx}{N}\right).$$

Evaluating this expression using our DFT coefficients gives us the following plot:



What Does This Have To Do with Linear Algebra?

If we define the function ω as

$$\omega = e^{-i2\pi/N}. \tag{18}$$

We can rewrite

$$F_k = \frac{1}{N} \sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} f_n e^{-i2\pi nk/N}. \tag{19}$$

as

$$F_k = \frac{1}{N} \sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} f_n \omega^{nk}. \tag{20}$$

Then we can write out the system of equations represented by this equation, giving us

$$F_{-\frac{N}{2}+1} = \frac{1}{N}\Big[f_{-\frac{N}{2}+1}\omega^{(-\frac{N}{2}+1)^2} + f_{-\frac{N}{2}+2}\omega^{(-\frac{N}{2}+2)(-\frac{N}{2}+1)} + \cdots$$

$$+ f_0\omega^{0(-\frac{N}{2}+1)} + \cdots + f_{-\frac{N}{2}}\omega^{(\frac{N}{2})(-\frac{N}{2}+1)}\Big]$$

$$F_{-\frac{N}{2}+2} = \frac{1}{N}\Big[f_{-\frac{N}{2}+1}\omega^{(-\frac{N}{2}+1)(-\frac{N}{2}+2)} + f_{-\frac{N}{2}+2}\omega^{(-\frac{N}{2}+2)^2} + \cdots$$

$$+ f_0\omega^{0(-\frac{N}{2}+2)} + \cdots + f_{-\frac{N}{2}}\omega^{(\frac{N}{2})(-\frac{N}{2}+2)}\Big]$$

$$\vdots$$

$$F_0 = \frac{1}{N}\Big[f_{-\frac{N}{2}+1}\omega^{(-\frac{N}{2}+1)(0)} + f_{-\frac{N}{2}+2}\omega^{(-\frac{N}{2}+2)(0)} + \cdots$$

$$+ f_0\omega^{(0)^2} + \cdots + f_{\frac{N}{2}}\omega^{(\frac{N}{2})(0)}\Big]$$

$$\vdots$$

$$F_{\frac{N}{2}-1} = \frac{1}{N}\Big[f_{-\frac{N}{2}+1}\omega^{(-\frac{N}{2}+1)(\frac{N}{2}-1)} + f_{-\frac{N}{2}+2}\omega^{(-\frac{N}{2}+2)(\frac{N}{2}-1)} + \cdots$$

$$+ f_0\omega^{0(\frac{N}{2}-1)} + \cdots + f_{-\frac{N}{2}}\omega^{(\frac{N}{2})(\frac{N}{2}-1)}\Big]$$

$$F_{\frac{N}{2}} = \frac{1}{N}\Big[f_{-\frac{N}{2}+1}\omega^{(-\frac{N}{2}+1)(\frac{N}{2})} + f_{-\frac{N}{2}+2}\omega^{(-\frac{N}{2}+2)(\frac{N}{2})} + \cdots$$

$$+ f_0\omega^{0(\frac{N}{2})} + \cdots + f_{-\frac{N}{2}}\omega^{(\frac{N}{2})^2}\Big].$$

We can rewrite this system of equations as a matrix expression, giving us

$$F = \frac{1}{N}Wf, \tag{21}$$

where W is the DFT matrix

$$\begin{bmatrix} \omega^{(-\frac{N}{2}+1)^2} & \omega^{(-\frac{N}{2}+2)(-\frac{N}{2}+1)} & \cdots & \omega^{0(-\frac{N}{2}+1)} & \cdots & \omega^{(\frac{N}{2}-1)(-\frac{N}{2}+1)} & \omega^{(-\frac{N}{2}+1)(\frac{N}{2})} \\ \omega^{(-\frac{N}{2}+1)(-\frac{N}{2}+2)} & \omega^{(-\frac{N}{2}+2)^2} & \cdots & \omega^{0(-\frac{N}{2}+2)} & \cdots & \omega^{(\frac{N}{2}-1)(-\frac{N}{2}+2)} & \omega^{(\frac{N}{2})(-\frac{N}{2}+2)} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ \omega^{(-\frac{N}{2}+1)(0)} & \omega^{(-\frac{N}{2}+2)(0)} & \cdots & \omega^{(0)^2} & \cdots & \omega^{(\frac{N}{2}-1)(0)} & \omega^{(\frac{N}{2})(0)} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ \omega^{(-\frac{N}{2}+1)(\frac{N}{2}-1)} & \omega^{(-\frac{N}{2}+2)(\frac{N}{2}-1)} & \cdots & \omega^{0(\frac{N}{2}-1)} & \cdots & \omega^{(\frac{N}{2}-1)^2} & \omega^{(\frac{N}{2})(\frac{N}{2}-1)} \\ \omega^{(-\frac{N}{2}+1)(\frac{N}{2})} & \omega^{(-\frac{N}{2}+2)(\frac{N}{2})} & \cdots & \omega^{0(\frac{N}{2})} & \cdots & \omega^{(\frac{N}{2}-1)(\frac{N}{2})} & \omega^{(\frac{N}{2})^2} \end{bmatrix}.$$

Back To Our Example

If we let

$$f = \begin{bmatrix} 0.7630 \\ -0.1205 \\ -0.0649 \\ 0.6133 \\ -0.2697 \\ -0.7216 \\ -0.0993 \\ 0.9787 \\ -0.5689 \\ -0.1080 \\ -0.3685 \\ 0.0293 \end{bmatrix}. \tag{22}$$

Using Matlab with the commands:

```
N=12;
f=[.7630;-.1205;-.0649;.6133;-.2697;-.7216;...
    -.0993;.9787;-.5689;-.1080;-.3685;.0293]
n=-5:6;
n=n';
k=-5:6;
H=n*k;
w=exp(-i*2*pi/12);
W=w.^H;
F=(1/N)*W*f
```

We see that our DFT coefficients are:

```
F =
    0.0684 - 0.1093i
   -0.1684 + 0.0685i
   -0.2143 - 0.0381i
   -0.0606 + 0.1194i
   -0.0418 - 0.0548i
    0.0052
   -0.0418 + 0.0548i
   -0.0606 - 0.1194i
   -0.2143 + 0.0381i
   -0.1684 - 0.0685i
    0.0684 + 0.1093i
    0.1066 + 0.0000i.
```

Applications

- Astronomical Data

  1754 Clairaut

  1800 Gauss

- Signal Processing
- Seismic Migration
- Digital Filtering

Conclusion

- Derive DFT
- Interpolate Data
- Applications