Arnold's Cat Map

Kyle Shaw

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Abstract

The purpose of this paper is to introduce the mapping known as Arnold's Cat Map, and to explain some of its properties using linear algebra.

1. Introduction

The mapping known as Arnold's Cat Map is named after the mathematician Vladimir I. Arnold, who first illustrated it using a diagram of a cat. Arnold's Cat map is defined as the mapping $\Gamma : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$\Gamma\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \bmod 1. \tag{1}$$

2. Modular Arithmetic

In order to describe Arnold's Cat Map, first we need to understand what modular arithmetic is.

The expression

$$A \bmod B$$

evaluates to the number N, such that

$$0 \leq N < B,$$

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and

$$A - N = k \times B,$$

where k is an integer. In other words, to find

$$A \bmod B$$

you add (or subtract) a multiple of B to A so that the result is in the interval [0,B). For example, 3 mod 10 is 3, -2.5 mod 5 is 2.5, and 15 mod 2 is 1. The expression

$$(X, Y) \bmod N$$

is the same as

$$(X \bmod N, Y \bmod N).$$

The effect of Γ is to map any point in the $\mathbb{R}^2$ plane to a point (x, y) in the unit square, where $0 \le x < 1$ and $0 \le y < 1$.

3. Shearing

We can rewrite (1) , using matrix factorization, in the form

$$\Gamma\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \bmod 1.$$

This factorization shows that Γ is composed of a shear in the x direction, followed by a shear in the y direction. Figure 1 shows the effect of these shears, done one at a time, on the unit square, which has been painted with a picture of a cat.

Because the determinant of the matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

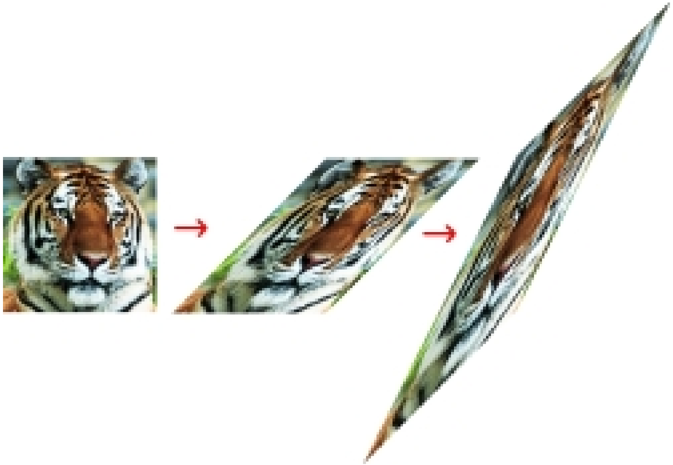

Figure 1: Shearing the cat (don't try this at home!)

is 1, any image in the $\mathbb{R}^2$ plane that has been transformed by this matrix will retain its original area. Therefore, the area of third picture in Figure 1 is the same as the area of the first picture. The effect of the mod 1 operation is to cut up the third image and reassemble it into the unit square, as seen in Figure 2. This is the completed mapping of Arnold's Cat Map performed once on the picture seen in Figure 1.

4. Order from Chaos?

After repeated applications of the cat map, we see some surprising things happen. The image seems to dissolve into a television-static like state, and then it inexplicably reforms into the original image! Figure 3 shows the effect of repeated applications of the cat map on a 124 by 124 pixel image. This may seem quite unusual at first, but it can actually be explained very reasonably. Let us consider the effect of the cat map on a single point in the $\mathbb{R}^2$ plane of the form $(a/n, b/n)$, where a, b, and

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Figure 2: Mod 1

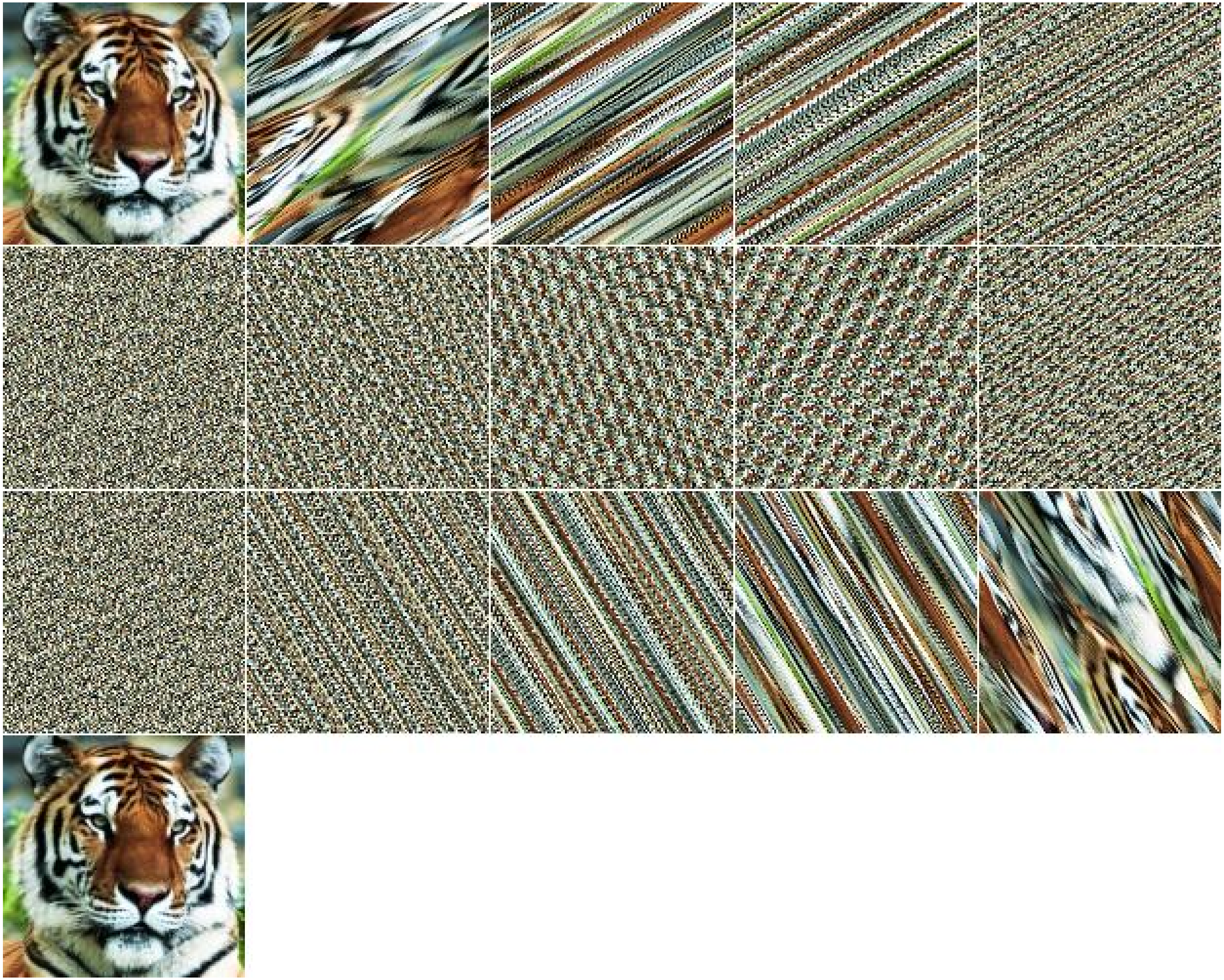

Figure 3: The full cycle

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n are integers, $0 \le a < n$ and $0 \le b < n$.

$$\begin{aligned}\Gamma\left(\begin{bmatrix} a/n \\ b/n \end{bmatrix}\right) &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a/n \\ b/n \end{bmatrix} \mod 1 \\ &= \begin{bmatrix} (a+b)/n \\ (a+2b)/n \end{bmatrix} \mod 1 \\ &= \begin{bmatrix} ((a+b)/n) \mod 1 \\ ((a+2b)/n) \mod 1 \end{bmatrix}\end{aligned}$$

We already know that Γ will map any point in the $\mathbb{R}^2$ plane to a point in the unit square. For a point in the unit square (but not on the upper or right edge) whose x and y coordinates are both rational numbers with lowest common denominator n, Γ will map that point to another point of the form $(c/n, d/n)$, with $0 \le c < n$ and $0 \le d < n$. Since there are only n^2 points of this form, it follows that a point of the form $(a/n, b/n)$ will be mapped back to itself after at most n^2 iterations of the cat map.

If the unit square has been colored in with a $N \times N$ image, we can consider it to have N^2 points of the form $(a/n, b/n)$ with colors assigned to them. After each iteration of the cat map to the image, each colored point (except for the point $(0,0)$) moves to a new position that was previously occupied by another colored point. Each point must return to its original position after no more than N^2 iterations of the cat map.

Let us say that if a point returns to its original position after P iterations of the cat map, it

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has a period of P. We can easily show that the period of the point $(0,0)$ is 1.

$$\begin{aligned}\Gamma\left(\begin{bmatrix}0\\0\end{bmatrix}\right) &= \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix} \mod 1. \\ &= \begin{bmatrix}0\\0\end{bmatrix} \mod 1 \\ &= \begin{bmatrix}0 \mod 1\\0 \mod 1\end{bmatrix} \\ &= \begin{bmatrix}0\\0\end{bmatrix}\end{aligned}$$

It turns out that the point $(0,0)$ is the only point with a period of one. For an $N \times N$ image, let the period of each point be represented by P_i. The period of the entire picture must then be the lowest common multiple of all the P_i. For example, if a picture in the unit square contained points with periods of 1, 3, 5, and 15 only, the number of iterations of the cat map required to return the image to its original state would be 15, as in the case of the 124 by 124 pixel image seen in Figure 3.

Figure 4 shows a graph of the periods of images up to 300 pixels by 300 pixels. You can see that the periods vary greatly from one image to another.

5. Streaks and Stripes

Another interesting behavior we see following applications of the cat map is the appearance of diagonal "streaks" in several of the intermediate images. In order to fully understand why this is happening, we have to think of the cat map in another way. Because of the mod 1 operation, the cat map is not a linear transformation. However, if the entire R^2 plane has been "tiled" with an image, and then we transform each point in the entire plane by multiplying it by the matrix

$$\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}$$

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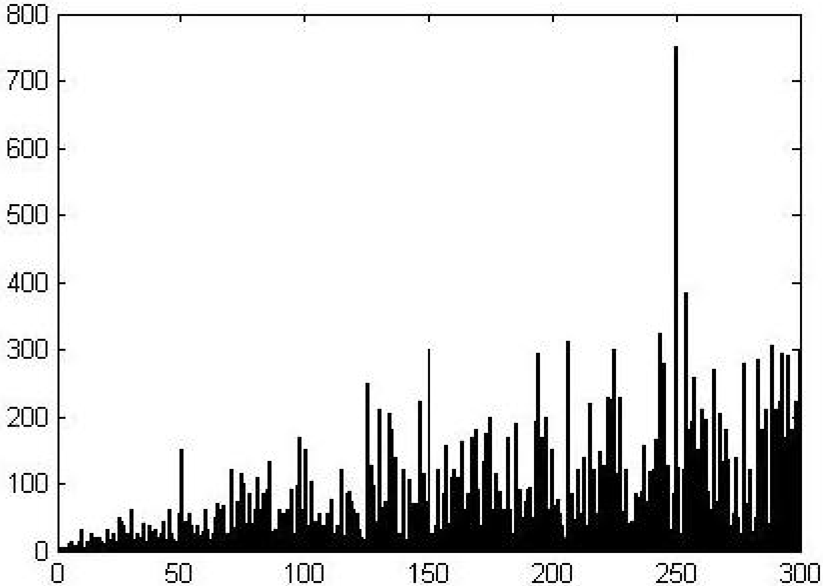

Figure 4: Period vs. size

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without using the mod 1 operation, we would see the same effect as if we had applied the cat map to just the unit square, with the mod 1 operation. We can explain the "streaks" by finding the eigenvectors and eigenvalues of the matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

We begin by finding the eigenvalues.

$$\begin{aligned} 0 &= \operatorname{Det}\left(\begin{bmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\right) \\ 0 &= 1 - 3\lambda + \lambda^2 \\ \lambda &= \frac{3 \pm \sqrt{5}}{2} \end{aligned}$$

We can see that the eigenvalues of the matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

are

$$\frac{3+\sqrt{5}}{2} \qquad \text{and} \qquad \frac{3-\sqrt{5}}{2}.$$

Next, we find the eigenvector corresponding to the eigenvalue

$$\frac{3+\sqrt{5}}{2}.$$

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We will call this vector $\mathbf{v_1}$.

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{v_1} = \frac{3+\sqrt{5}}{2} \mathbf{v_1}$$

$$\left(\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - \frac{3+\sqrt{5}}{2} \mathbf{I} \right) \mathbf{v_1} = \mathbf{0}$$

$$\begin{bmatrix} -\frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \mathbf{v_1} = \mathbf{0}$$

To eliminate the first entry of the second row, first we divide it by the first entry in the first row. In this case the result is

$$-\frac{2}{1+\sqrt{5}}.$$

Then we multiply the first row by this result and subtract it from the second row.

$$\begin{bmatrix} -\frac{1+\sqrt{5}}{2} & 1 \\ 1 - \frac{-(1+\sqrt{5})}{2}\left(-\frac{2}{1+\sqrt{5}}\right) & \frac{1-\sqrt{5}}{2} - 1\left(-\frac{2}{1+\sqrt{5}}\right) \end{bmatrix} \mathbf{v_1} = \mathbf{0}$$

$$\begin{bmatrix} -\frac{1+\sqrt{5}}{2} & 1 \\ 0 & 0 \end{bmatrix} \mathbf{v_1} = \mathbf{0}$$

Therefore the first eigenvector can be any point (x, y) such that

$$0 = -\frac{1+\sqrt{5}}{2} x + y$$

$$y = \frac{1+\sqrt{5}}{2} x$$

For simplicity, we choose the point with the x coordinate of 1.

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}$$

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Now that we have found the first eigenvector, we will find the eigenvector that corresponds to the eigenvalue

$$\frac{3-\sqrt{5}}{2},$$

and we will call this vector $\mathbf{v_2}$.

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{v_2} = \frac{3-\sqrt{5}}{2}\mathbf{v_2}$$

$$\left(\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - \frac{3-\sqrt{5}}{2}\mathbf{I} \right) \mathbf{v_2} = \mathbf{0}$$

$$\begin{bmatrix} \frac{-1+\sqrt{5}}{2} & 1 \\ 1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \mathbf{v_2} = \mathbf{0}$$

Again we want to eliminate the first entry of the second row, so we divide it by the first entry in the first row. Here the result is

$$\frac{2}{-1+\sqrt{5}}.$$

Now we multiply the first row by this result and subtract it from the second row.

$$\begin{bmatrix} \frac{-1+\sqrt{5}}{2} & 1 \\ 1 - \frac{-1+\sqrt{5}}{2}\left(\frac{2}{-1+\sqrt{5}}\right) & \frac{1+\sqrt{5}}{2} - 1\left(\frac{2}{-1+\sqrt{5}}\right) \end{bmatrix} \mathbf{v_2} = \mathbf{0}$$

$$\begin{bmatrix} \frac{-1+\sqrt{5}}{2} & 1 \\ 0 & 0 \end{bmatrix} \mathbf{v_2} = \mathbf{0}$$

The second eigenvector can be any point (x, y) such that

$$0 = \frac{-1+\sqrt{5}}{2}x + y$$

$$y = -\frac{-1+\sqrt{5}}{2}x$$

Again for simplicity, we choose the point with the x coordinate of 1.

$$\mathbf{v_2} = \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

Thus, the eigenvalues and corresponding eigenvectors of the matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

are

$$\lambda_1 = \frac{3+\sqrt{5}}{2} \approx 2.6180\ldots, \quad \mathbf{v_1} = \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix} \approx \begin{bmatrix} 1 \\ 1.6180 \end{bmatrix}$$

and

$$\lambda_2 = \frac{3-\sqrt{5}}{2} \approx 0.38196\ldots, \quad \mathbf{v_2} = \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix} \approx \begin{bmatrix} 1 \\ -0.6180 \end{bmatrix}.$$

The “streaks” that appear are in the directions of the two eigenvectors of the matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Figure 5 shows the directions of the eigenvectors $\mathbf{v_1}$ and $\mathbf{v_2}$. As each vector representing a point in the unit square is multiplied by the matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},$$

it moves away from its original position in the general direction of the vector

$$\begin{bmatrix} 1 \\ 1.6810 \end{bmatrix}.$$

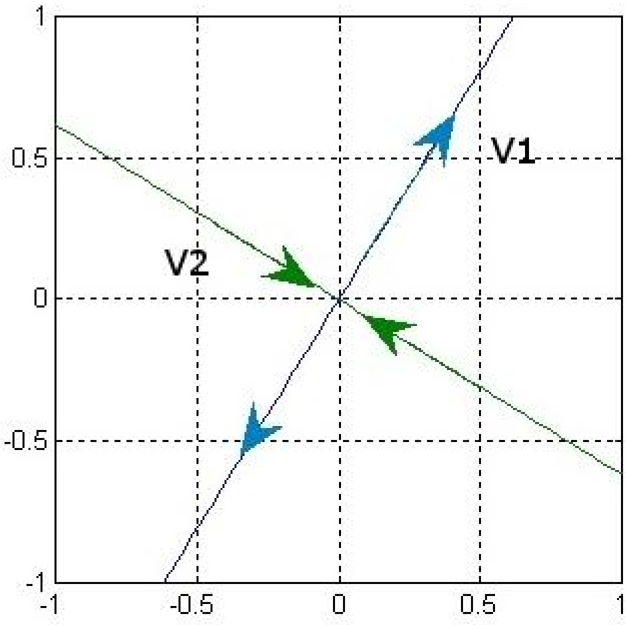

Figure 5: Eigenvector directions

Now let us consider a point a in the unit square that is assigned a specific color. If the point a has period P, then somewhere in the "tiled" plane there is another point b that will arrive at the original position of point a after the entire plane has been transformed P times. Each time the vector representing the point b is multiplied by the matrix above, it moves closer to the original position of the point a, finally arriving there after P multiplications. Upon the final transformation of the point b, it moves to the original position of the point a in the direction of the eigenvector $\mathbf{v_2}$. This is the cause of the "streaks" that appear near the end of the cycle, which are in the direction of the eigenvector $\mathbf{v_2}$.

6. Apparitions

Another peculiar thing occurs when the cat map is applied to some larger images. An image that is 768 pixels by 768 pixels has a period of 192. Figure 6 shows what happens after 96 iterations of the cat map on a 768 pixel by 768 pixel image.

In the background of the transformed image we can see a faint, ghostly likeness of the original image. There are also sections of the original image that have been moved to new locations in the transformed image. We can explain the appearance of the original image in the background in the following way. If the unit square has been painted with a 768 by 768 pixel image, it consists of colored points of the form $(i/768, j/768)$, where i and j are integers, $0 \le i < 768$ and $0 \le j < 768$. When i and j are multiples of 2, then the points reduce to the form $(p/384, q/384)$. These points are the same points we would have if we painted the unit square with a 384 by 384 pixel image. The period of a 384 by 384 pixel image is exactly 96. So after 96 iterations of the cat map on a 768 by 768 pixel image, all the points in the image that have the same coordinates as points in a 384 by 384 pixel image have returned to their original positions. This is why we see the ghostly form of the original 768 by 768 pixel image reappearing after 96 iterations of the cat map.

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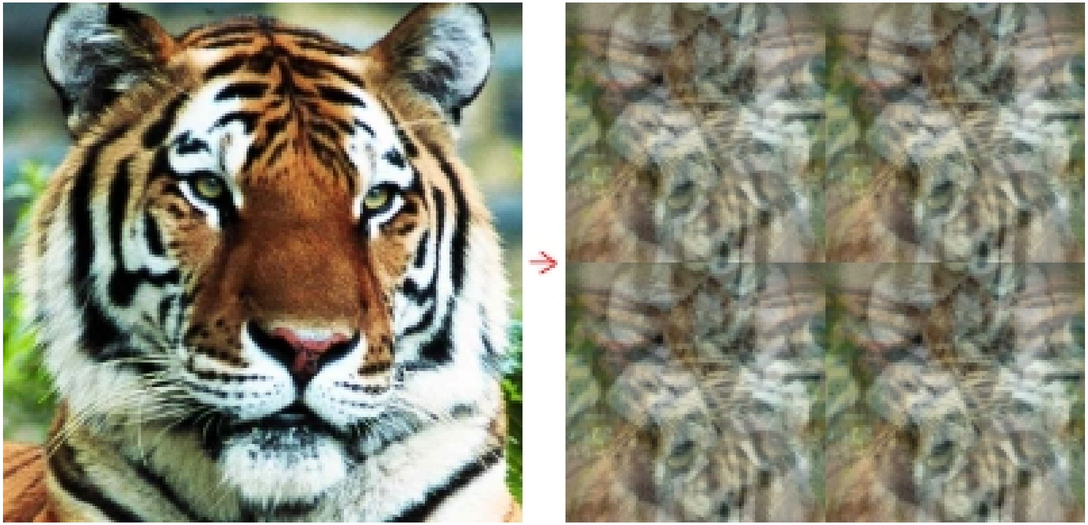

Figure 6: The ghost and the darkness

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