Section 11.1
Systems of Equations

Definition

A linear equation in two variables has the form

$$ax + by = c$$

A solution of the linear equation is any ordered pair $(x, y) = (x_0, y_0)$ that satisfies the equation. If x_0 and y_0 are substituted into the equation. then $ax_0 + by_0 = c$ is a true statement.

Systems of Equations in Two Variables

In many applications it is often necessary to use more than one equation to solve a problem.

In these cases we use a system of equations.

To solve a system of equations, we find an ordered pair that satisfies each equation in the system.

Definition

A system of linear equations in two variables has the form

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

A solution of the system of linear equations is any ordered pair $(x, y) = (x_0, y_0)$ that satisfies both the equations. If x_0 and y_0 are substituted into the equations, then $ax_0 + by_0 = c$ and $dx_0 + ey_0 = f$ are true statements.

Definition

A system of equations that has at least one solution is called consistent.

A system of equations that has no solutions is called inconsistent.

Suppose the perimeter of a garden is 94 feet, and the length is 4 more than the width.

We get the following system of equations, where l is the length and w is the width.

$$2l + 2w = 94$$
$$l = w + 4$$

To solve this system means to find the values for l and w that satisfy both equations.

What happens if we graph each equation?

We let y be the length and x be the width.

$$2y+2x=94$$
$$y=x+4$$

To use our calculators, we use the form $y=mx+b$

$$y=-x+47$$
$$y=x+4$$

If we chose an appropriate window size, we see that the solution to the system occurs at the point of intersection.

We can estimate this point of intersection on a piece of graph paper or use our calculator.

Let's use our calculator.

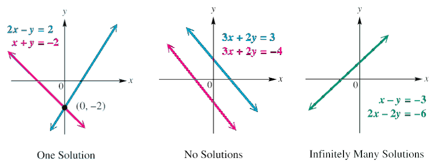
It appears that the solution is $(w,l)=(21.5,25.5)$

Does this ordered pair satisfy each equation?

Graphing Systems of Equations

- Graph each of the equations.
- The solutions of the system are given by the points of intersection.
- If the graphs do not intersect, then there are no solutions and we say the equations are inconsistent.
- Any system with exactly one solution is said to be independent.
- If the graphs are identical, then we say every point on the graph is a solution and we say the equations are dependent.

Linear System in Two Variables



- Three possible solutions to a linear system in two variables:
  1. One solution: coordinates of a point,
  2. No solutions: **inconsistent** case,
  3. Infinitely many solutions: **dependent** case.

The Substitution Method

- Given a system of two equations in the variables x and y.
- Solve one of the equations for y.
- Substitute this expression into the other equation in place of y then solve the resulting equation for x.
- Go back to one of the original equations and use this value for x to solve for the variable y.
- Check the solution.

Solve the following system by substitution.

$$x + y = 6$$
$$y = x - 2$$

Solve the following system.

$$x + y = 6$$
$$x + 2y = 12$$

In this case, first solve one of the equations for y

Solve the following system.

$$2x = 3y - 6$$
$$x = 3y$$

In this case, substitute for x instead of y then solve.

Solve the following system.

$$x + y = 6$$
$$x + y = 12$$

Solve the following system.

$$2x = 3y - 6$$
$$4x = 6y - 12$$

Solve the following system.

$$2x - y = 3$$
$$y = \frac{1}{2}x + 1$$

Solve the following system.

$$x + 2y = 10$$
$$3x + 4y = 8$$

Which variable should you solve for first?

Solve the following system.

$$2x + 2y = 10$$
$$3x - 4y = 8$$

Which variable should you solve for first?

In many cases, the substitution method is difficult to use. In these cases try to use another method.

One such method is called the ELIMINATION METHOD

The Elimination Method

- Given a system of two equations in the variables x and y.
- Rewrite each equation in the form $Ax+By=C$
- Add a multiple of one equation to the other equation so that one of the variables is eliminated.
- Solve the resulting equation for the variable.
- Go back to one of the original equations and use this value to solve for the other variable.
- Check the solution.

Solve the following system.

$$x + y = 6$$
$$x + 2y = 12$$

Solve the following system by elimination.

$$3x + 2y = 6$$
$$6x - 3y = 4$$

Solve the following system.

$$2x - 3y = -6$$
$$4x + 2y = 4$$

Solve the following system.

$$x + 2y = 10$$
$$3x + 4y = 8$$

Solve the following system.

$$1.2x + 2.5y = 10$$
$$x - 4y = 8$$

A problem from ancient China (152 BC)

Suppose there are some rabbits and pheasants in a cage. There are 35 heads and 94 feet. Find the number of rabbits and the number of pheasants.

A problem from Wonderland

In Lewis Carroll's *Through the Looking Glass*, Tweedledum says to Tweedledee, "The sum of your weight and twice mine is 361 pounds." Then Tweedledee says to Tweedledum, "Contrariwise, the sum of your weight and twice mine is 362 pounds." Find the weight of each.

Mixture Problems

Sunflower seed costs $1.00 per pound. Rolled oats cost $1.35 per pound. How many pounds of each seed would you need to make 50 lbs of a mixture that costs $1.14 per pound?

Mixture Problems

One solution is 15% salt, and second solution is 20%. How many liters of each solution must be mixed to obtain 50 liters of a 16% salt solution?

Example

A jeweler wishes to make a 60 oz mixture that is two-thirds pure gold. She has two stocks of gold alloy, the first stock contains three-fourths pure gold and the second stock is five-twelfths pure gold. How many ounces of each stock does she need?

Definition

A linear equation in three variables has the form

$$ax + by + cz = d$$

A solution of the linear equation is any ordered triple $(x, y, z) = (x_0, y_0, z_0)$ that satisfies the equation. If x_0, y_0 and z_0 are substituted into the equation. then $ax_0 + by_0 + cz_0 = d$ is a true statement.

Definition

A system of linear equations in three variables has the form

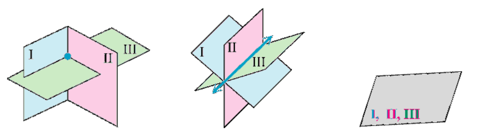
$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

A solution of the system of linear equations is any ordered triple (x_0, y_0, z_0) that satisfies all the equations.

- Solutions of systems with 3 variables with linear equations of the form

  $Ax + By + Cz = D$ (a plane in 3-D space)

  are called **ordered triples** (x, y, z).
- Possible solutions:



One Solution　Infinitely Many Solutions　Infinitely Many Solutions

Solve the following system.

$$3x + 2y - z = -3$$
$$5y - 2z = 2$$
$$5z = 20$$

We start by solving for z. In this case $z = 4$

Then we substitute $z = 4$ back into the second equation to solve for y.

$$5y - 2(4) = 2$$
$$5y - 8 = 2$$
$$5y = 10$$
$$y = 2$$

Finally, we substitute $y = 2$ and $z = 4$ back into the first equation to solve for x.

$$3x + 2y - z = -3$$
$$3x + 2(2) - (4) = -3$$
$$3x = -3$$
$$x = -1$$

So the solution to this system is the ordered triple $(x, y, z) = (-1, 2, 4)$.

$$\begin{array}{rrrrrrrl} x & + & ay & + & bz & = & c & \textbf{(1)} \\ & & y & + & dz & = & e & \textbf{(2)} \\ & & & & z & = & f & \textbf{(3)} \end{array}$$

A system of equations is said to be in upper triangular form provided that x appears in no equation after the first and y appears in no equation after the second.

Solve the following system.

$$\begin{aligned} x + 2y + z &= 3 \\ 2x + y + z &= 16 \\ x + y + 2z &= 9 \end{aligned}$$

BIG IDEA: Use Gaussian Elimination to transform this system into upper triangular form.

Elementary Row Operations

1. Multiply an equation by a constant.
2. Interchange the order of any two equations
3. Add a multiple of one equation to any other equation in the system.

$$\begin{cases} x+2y+z=3 \\ 2x+y+z=16 \\ x+y+2z=9 \end{cases}$$

To eliminate x from equation (2) we add -2 times the first equation to the second equation

$$\begin{cases} x+2y+z=3 \\ \quad -3y-z=10 \\ x+y+2z=9 \end{cases}$$

$$\begin{cases} x+2y+z=3 \\ \quad -3y-z=10 \\ x+y+2z=9 \end{cases}$$

To eliminate x from equation (3) we add -1 times the first equation to the third equation

$$\begin{cases} x+2y+z=3 \\ \quad -3y-z=10 \\ \quad -y+z=6 \end{cases}$$

We can interchange equations (2) and (3)

$$\begin{cases} x+2y+z=3 \\ \quad -y+z=6 \\ \quad -3y-z=10 \end{cases}$$

We can multiply equation (2) by -1

$$\begin{cases} x+2y+z=3 \\ \quad y-z=-6 \\ \quad -3y-z=10 \end{cases}$$

We can add 3 times equation (2) to (3)

$$\begin{cases} x+2y+z=3 \\ y-z=-6 \\ -4z=-8 \end{cases}$$

We can divide equation (3) by -4

$$\begin{cases} x+2y+z=3 \\ y-z=-6 \\ z=2 \end{cases}$$

$$\begin{cases} x+2y+z=3 \\ y-z=-6 \\ z=2 \end{cases}$$

Using back substitution we get

$z=2$

$y-2=-6$ so $y=-4$

$x+2(-4)+2=3$

$x+(-6)=3$ so $x=9$

the solution to the system is $(x,y,z)=(2,-4,9)$

Solve the following system. $\begin{cases} x+y+z=-1 \\ -2x+y+z=-2 \\ 3x+6y+6z=5 \end{cases}$

$2E_1+E_2$ $\begin{cases} x+y+z=-1 \\ -2x+y+z=-2 \\ 3x+6y+6z=5 \end{cases}$ $\begin{cases} x+y+z=-1 \\ 3y+3z=-4 \\ 3x+6y+6z=5 \end{cases}$

$-3E_1+E_3$ $\begin{cases} x+y+z=-1 \\ 3y+3z=-4 \\ 3x+6y+6z=5 \end{cases}$ $\begin{cases} x+y+z=-1 \\ 3y+3z=-4 \\ 3y+3z=8 \end{cases}$

$$
\begin{cases} x+y+z=-1 \\ 3y+3z=-4 \\ 3y+3z=8 \end{cases} \quad -E_2+E_3 \qquad \begin{cases} x+y+z=-1 \\ 3y+3z=-4 \\ 0=12 \end{cases}
$$

The equation $0=12$ is a contradiction!

This means there are no solutions to this system.

Solve the following system.

$$
\begin{cases} 2x+y+z=3 \\ 2z=8 \end{cases}
$$

If we use the elimination method we can solve for z to get $z=4$

Using back-substitution we get the equation

$$2x+y+4=3$$

If we solve for y we get the equation

$$y=-2x-1$$

In this case there are an infinite number of solutions to the system. We describe any solution as follows

$z=4,\ \ y=-2x-1,\ \ x=$ any number

We usually describe the solutions as an ordered triple, so any solution has the form

$(x,-2x-1,4)$ where $x=$ any number

Solve the following system.

$-2E_2 + E_1 \quad \begin{cases} 2x+4y+2z=3 \\ x+2y+z=1 \end{cases}$

If we use the elimination method we can add -2 times the second equation to the first equation.

$\begin{cases} 2x+4y+2z=3 \\ -2x-4y-2z=-2 \end{cases}$

so $0=1$

so there are no solutions!

Example

In a triangle, the largest angle is 70° more than the smallest angle and twice as large as the other angle. Use a system of equations to find the angles.

Let x be the largest angle, y be the second angle, and z be the smallest angle.

$\begin{cases} x+y+z=180 \\ x=z+70 \\ x=2y \end{cases}$

Solve this system.