

Section 12.4  
Mathematical Induction

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12.4 Mathematical Induction

- Mathematical induction is used to prove statements claimed true for every positive integer  $n$ .

For example, the summation rule

$$1+2+3+\dots+n = \frac{n(n+1)}{2}$$

is true for each integer  $n \geq 1$ .

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12.4 Mathematical Induction

Label the statement  $S_n$ .

$$S_n : 1+2+3+\dots+n = \frac{n(n+1)}{2}$$

For any one value of  $n$ , the statement can be verified to be true.

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$n = 1, S_1: 1 = \frac{1(1+1)}{2}$  true since  $1=1$   
 $n = 2, S_2: 1 + 2 = \frac{2(2+1)}{2}$  true since  $3=3$   
 $n = 3, S_3: 1 + 2 + 3 = \frac{3(3+1)}{2}$  true since  $6=6$

To show  $S_n$  is true for every  $n$  requires mathematical induction.

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**Principle of Mathematical Induction**

Let  $S_n$  be a statement concerning the positive integer  $n$ . Suppose that

- $S_1$  is true;
- For any positive integer  $k, k \leq n$ , if  $S_k$  is true, then  $S_{k+1}$  is also true.

Then,  $S_n$  is true for every positive integer  $n$ .

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**Proof by Mathematical Induction**

**Step 1** Prove that the statement is true for  $n = 1$ .

**Step 2** Show that for any positive integer  $k$ , if  $S_k$  is true, then  $S_{k+1}$  is also true.

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**Example** Let  $S_n$  be the statement

$$S_n: 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

Prove that  $S_n$  is true for every positive integer  $n$ .

**Solution** The proof uses mathematical induction.

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**Solution Step 1** Show that the statement is true when  $n = 1$ .  $S_1$  is the statement

$$1 = \frac{1(1+1)}{2}$$

which is true since both sides equal 1.

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**Solution Step 2** Show that if  $S_k$  is true then  $S_{k+1}$  is also true. Start with  $S_k$

$$1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2}$$

and assume it is a true statement. Add  $k + 1$  to each side

$$1 + 2 + 3 + \dots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$


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**Solution Step 2**

$$\begin{aligned} 1 + 2 + 3 + \dots + k + (k + 1) &= \frac{k(k + 1)}{2} + (k + 1) \\ &= (k + 1) \left( \frac{k}{2} + 1 \right) \\ &= (k + 1) \left( \frac{k + 2}{2} \right) \end{aligned}$$

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**Solution Step 2**

$$\begin{aligned} 1 + 2 + 3 + \dots + k + (k + 1) &= (k + 1) \left( \frac{k + 2}{2} \right) \\ &= (k + 1) \left( \frac{(k + 1) + 1}{2} \right) \end{aligned}$$

This is the statement for  $n = k + 1$ . It has been shown that if  $S_k$  is true then  $S_{k+1}$  is also true. By mathematical induction  $S_n$  is true for all positive integers  $n$ .

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**Generalized Principle of Mathematical Induction**

Let  $S_n$  be a statement concerning the positive integer  $n$ . Suppose that

**Step 1**  $S_j$  is true;

**Step 2** For any positive integer  $k, k \geq j$ , if  $S_k$  implies

$$S_{k+1}.$$

Then,  $S_n$  is true for all positive integers  $n \geq j$ .

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**Example** Let  $S_n$  represent the statement  
$$S_n : 2^n > 2n + 1$$
  
Show that  $S_n$  is true for all values of  $n \geq 3$ .  
**Solution** Since the statement is claimed to be true for values of  $n$  beginning with 3 and not 1, the proof uses the generalized principle of mathematical induction.

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**Solution Step 1** Show that  $S_n$  is true when  $n = 3$ .  $S_3$  is the statement  
$$2^3 > 2 \cdot 3 + 1$$
  
which is true since  $8 > 7$ .

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**Solution Step 2** Show that  $S_k$  implies  $S_{k+1}$  for  $k \geq 3$ .  
Assume  $S_k$   
$$2^k > 2k + 1$$
  
is true. Multiply each side by 2, giving  
$$2 \cdot 2^k > 2(2k + 1)$$
  
or 
$$2^{k+1} > 4k + 2$$
  
or, equivalently 
$$2^{k+1} > 2(k + 1) + 2k .$$

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**Solution Step 2** Since  $k \geq 3$ , then  $2k > 1$  and it follows that

$$2^{k+1} > 2(k+1) + 2k > 2(k+1) + 1$$

or

$$2^{k+1} > 2(k+1) + 1$$

which is the statement  $S_{k+1}$ . Thus  $S_k$  implies  $S_{k+1}$  and, by the generalized principle,  $S_n$  is true for all  $n \geq 3$ .

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