

Section 12.5  
The Binomial Theorem

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Simplify  $(x + y)^2$

$$(x + y)^2 = (x + y)(x + y)$$
$$= x^2 + xy + xy + y^2$$
$$= x^2 + 2xy + y^2$$

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Simplify  $(x + y)^3$

$$(x + y)^3 = (x + y)(x + y)(x + y)$$
$$= (x^2 + 2xy + y^2)(x + y)$$
$$= x^3 + 2x^2y + xy^2$$
$$+ x^2y + 2xy^2 + y^3$$
$$= x^3 + 3x^2y + 3xy^2 + y^3$$

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Simplify  $(x + y)^4$

$$\begin{aligned}(x + y)^4 &= (x + y)^3 (x + y) \\ &= (x^3 + 3x^2y + 3xy^2 + x^3)(x + y) \\ &= x^4 + 3x^3y + 3x^2y^2 + xy^3 \\ &\quad + x^3y + 3x^2y^2 + 3xy^3 + y^4 \\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\end{aligned}$$

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What about  $(x + y)^n$

- There are  $n + 1$  terms in the polynomial
- The expansion begins with  $x^n$  and ends with  $y^n$
- The sum of the exponents in every term is  $n$
- The exponents of  $x$  decrease by 1 from left to right
- The exponents of  $y$  increase by 1 from left to right

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Break into a group of four members and complete the Pascal's Triangle Activity

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### Pascal's Triangle

						Row
		1				0
		1	1			1
		1	2	1		2
		1	3	3	1	3
	1	4	6	4	1	4
1	5	10	10	5	1	5

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### Pascal's Triangle

- Each row of the triangle begins with a 1 and ends with a 1.
- Each number in the triangle that is not a 1 is the sum of the two numbers directly above it (one to the right and one to the left.)
- Numbering the rows of the triangle 0, 1, 2, ... starting at the top, the numbers in row  $n$  are the coefficients of  $x^n, x^{n-1}y, x^{n-2}y^2, x^{n-3}y^3, \dots, y^n$  in the expansion of  $(x + y)^n$ .

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### $n$ -Factorial

$$n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$$

$$0! = 1$$

$$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$$

$$7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 5040$$

Evaluate  $10!$

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The Binomial Coefficient

Let  $n$  and  $k$  be nonnegative integers,  $k \leq n$ .

The binomial coefficient  $\binom{n}{k}$  is defined

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(1 \cdot 2)(1 \cdot 2 \cdot 3)} = \frac{4 \cdot 5}{1 \cdot 2} = 10$$

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Evaluate the following

$$\binom{6}{3} = \qquad \binom{3}{0} =$$

$$\binom{7}{5} = \qquad \binom{4}{4} =$$

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The symbol  $\binom{n}{r}$  is also denoted by  ${}_n C_r$  and is read " $n$  choose  $r$ "

The values of  $\binom{n}{r}$  are the values of the  $n^{\text{th}}$  row of Pascal's Triangle!

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${}_0C_0$	0
${}_1C_0$ ${}_1C_1$	1
${}_2C_0$ ${}_2C_1$ ${}_2C_2$	2
${}_3C_0$ ${}_3C_1$ ${}_3C_2$ ${}_3C_3$	3
${}_4C_0$ ${}_4C_1$ ${}_4C_2$ ${}_4C_3$ ${}_4C_4$	4
${}_5C_0$ ${}_5C_1$ ${}_5C_2$ ${}_5C_3$ ${}_5C_4$ ${}_5C_5$	5
1	0
1   1	1
1   2   1	2
1   3   3   1	3
1   4   6   4   1	4
1   5   10   10   5   1	5

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$\binom{3}{0}$  is the first value in the third row

$\binom{3}{1}$  is the second value in the third row

$\binom{3}{2}$  is the third value in the third row

$\binom{n}{r-1}$  is the  $r^{\text{th}}$  value in the  $n^{\text{th}}$  row

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**The Binomial Theorem**

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots$$

$$+ \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

The  $r^{\text{th}}$  term of  $(a+b)^n$  is given by

$$\binom{n}{r-1}a^{n-r+1}b^{r-1}$$

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Use the binomial theorem to find  $(a+b)^9$

$$\begin{aligned} (a+b)^9 &= \binom{9}{0}a^9 + \binom{9}{1}a^8b + \binom{9}{2}a^7b^2 + \binom{9}{3}a^6b^3 \\ &\quad + \binom{9}{4}a^5b^4 + \binom{9}{5}a^4b^5 + \binom{9}{6}a^3b^6 \\ &\quad + \binom{9}{7}a^2b^7 + \binom{9}{8}ab^8 + \binom{9}{9}b^9 \end{aligned}$$

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If the  $r^{\text{th}}$  term of  $(a+b)^n$  is given by

$$\binom{n}{r-1}a^{n-r+1}b^{r-1}$$

Find the 15<sup>th</sup> term of  $(a+b)^{20}$

$$n = 20, \quad r = 15$$

$$\begin{aligned} \text{so we have } \binom{20}{14}a^{20-14+1}b^{14-1} &= 38760a^{20-14+1}b^{14-1} \\ &= 38760a^7b^{13} \end{aligned}$$

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Find the 15<sup>th</sup> term of  $\left(x^2 - \frac{1}{x}\right)^{18}$

$$n = 18, \quad r = 15$$

$$\begin{aligned} \text{so we have } \binom{n}{r-1}a^{n-r+1}b^{r-1} &= \binom{18}{15-1}\left(x^2\right)^{18-15+1}\left(\frac{-1}{x}\right)^{15-1} \\ &= \binom{18}{14}\left(x^2\right)^4\left(\frac{-1}{x}\right)^{14} = \binom{18}{14}x^8\frac{1}{x^{14}} \\ &= 3060\frac{1}{x^6} = 3060x^{-6} \end{aligned}$$

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